Corinne Touati, INRIA

Robot Cockroach Tests Insect Decision-Making Behavior
Optimality of Multi-user system

- Optimality of a **single user**

Utility vs Parameter graph with an optimal point marked.
Situation with *multiple* users

- Good for user 2
- Good for user 1
Definition.

A point is Pareto optimal if it cannot be strictly dominated.
Definition.

A point is Pareto optimal if it cannot be strictly dominated.
Definition.

A point is Pareto optimal if it cannot be strictly dominated.
Optimality of Multi-user system

Definition.
A point is Pareto optimal if it cannot be strictly dominated.
Definition.
A point is Pareto optimal if it cannot be strictly dominated.

![Graph showing Pareto optimality](image_url)
Definition.

A point is Pareto optimal if it cannot be strictly dominated.
Definition.

A point is Pareto optimal if it cannot be strictly dominated.
Definition.
A point is Pareto optimal if it cannot be strictly dominated.
Definition.
A point is Pareto optimal if it cannot be strictly dominated.
Definition.
A point is Pareto optimal if it cannot be strictly dominated.

Utility of user 2
Utility of user 1

Optimality of Multi-user system
Definition.

A point is Pareto optimal if it cannot be strictly dominated.
## Cooperating or being selfish?

<table>
<thead>
<tr>
<th><strong>Cooperative games</strong></th>
<th><strong>Non-cooperative games</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Institution setting rules and penalties to enforce them</td>
<td>Individual behavior converge (or not) to an equilibrium</td>
</tr>
</tbody>
</table>

**Example:** Routing intersection:

- **Cooperative approach:** set of roadsigns (traffic lights, “stop signs”...) enforced by the police
- **Non-cooperative approach:** everyone tries to cross it as quickly as possible
Outline

1. Non-cooperative optimization
   - Nash Equilibria
   - Braess Paradoxes
   - Dynamic games
   - Other equilibria

2. Cooperative Games
   - Definitions of fairness
   - Examples

3. Conclusion
   - Other yet interesting topics...
   - Cooperation versus Selfishness: A last example
1. Non-cooperative optimization
   - Nash Equilibria
   - Braess Paradoxes
   - Dynamic games
   - Other equilibria

2. Cooperative Games
   - Definitions of fairness
   - Examples

3. Conclusion
   - Other yet interesting topics...
   - Cooperation versus Selfishness: A last example
Definition
In a non-cooperative setting, each player makes a decision so as to maximize its own return.

Nash equilibria
In a Nash equilibrium, no player has incentive to unilaterally modify his strategy.

\[ s^* \] is a Nash equilibrium iff:

\[
\forall p, \forall s_p, u_p(s_1^*, \ldots, s_p^*, \ldots s_n^*) \geq u_p(s_1^*, \ldots, s_p, \ldots, s_n^*)
\]
Nash Equilibria: definition (cont.)

Pros

▶ Intuitive

Cons

▶ No guaranty of existence / unicity
▶ difficult to compute analytically (fixed points)
▶ usually not Pareto optimal
Nash Equilibria: definition (cont.)

Pros
▶ Intuitive
▶ Easy to implement

Cons

Corinne Touati (INRIA)  Introduction to Game Theory  Non-cooperative optimization 7 / 33
Nash Equilibria: definition (cont.)

**Pros**
- Intuitive
- Easy to implement

**Cons**
- No guaranty of existence / unicity
Nash Equilibria: definition (cont.)

Pros
- Intuitive
- Easy to implement

Cons
- No guaranty of existence / unicity
- Difficult to compute analytically (fixed points)
Nash Equilibria: definition (cont.)

Pros
▶ Intuitive
▶ Easy to implement

Cons
▶ No guaranty of existence / unicity
▶ difficult to compute analytically (fixed points)
▶ usually not Pareto optimal
Nash equilibria: Applications

Various contexts:

- **Load balancing systems**
  Users decide which server to send their request so as to minimize their average delay.

- **Wireless systems**
  Users decide what power to use so as to maximize a compromise between the transfer rate and the battery usage.

- **Pricing systems**
  Providers choose their prices so as to maximize their revenue, which is a function of their charged price and their infrastructure cost and market share.

- **Queuing systems**
  Users optimize their “power” defined as the ratio of their throughput and their expected delay.
Nash equilibria: Applications

Various contexts:

- **Load balancing systems**
  Users decide which server to send their request so as to minimize their average delay.

- **Wireless systems**
  Users decide what power to use so as to maximize a compromise between the transfer rate and the battery usage.
Various contexts:

- **Load balancing systems**
  Users decide which server to send their request so as to minimize their average delay.

- **Wireless systems**
  Users decide what power to use so as to maximize a compromise between the transfer rate and the battery usage.

- **Pricing systems**
  Providers choose their prices so as to maximize their revenue, which is a function of their charged price and their infrastructure cost and market share.
Nash equilibria: Applications

Various contexts:

▶ **Load balancing systems**
Users decide which server to send their request so as to minimize their average delay.

▶ **Wireless systems**
Users decide what power to use so as to maximize a compromise between the transfer rate and the battery usage.

▶ **Pricing systems**
Providers choose their prices so as to maximize their revenue, which is a function of their charged price and their infrastructure cost and market share.

▶ **Queuing systems**
Users optimize their “power” defined as the ratio of their throughput and their expected delay.
Nash equilibria: Application to scheduling of bg-of-task applications

Two computers / two applications

\[ M \]

\[ b_1 = 1 \]
\[ b_2 = 2 \]

\[ w_1 = 2 \]
\[ w_2 = 1 \]

Cooperative Approach:
Application \( i \) is processed exclusively on computer \( i \).

\[ \alpha(\text{coop})_1 = \alpha(\text{coop})_2 = 1 \]

Computation time

\[ \text{Slave 1} \]
\[ \text{Slave 2} \]

Communication time

Non-Cooperative Approach:
\[ \alpha(\text{nc})_1 = 3 \]
\[ \alpha(\text{nc})_2 = 4 \]

\[ \text{Slave 1} \]
\[ \text{Slave 2} \]
Nash equilibria: Application to scheduling of bg-of-task applications

Two computers / two applications

Cooperative Approach:
Application $i$ is processed exclusively on computer $i$. 
$\alpha_i^{(coop)} = 1$.

Non-Cooperative Approach:
$\alpha_i^{(nc)} = 3$. 

Corinne Touati (INRIA)
Introduction to Game Theory
Non-cooperative optimization
Nash equilibria: Application to scheduling of bg-of-task applications

Two computers / two applications

Cooperative Approach:
Application $i$ is processed exclusively on computer $i$. $\alpha_1^{(coop)} = \alpha_2^{(coop)} = 1$

Non-Cooperative Approach:
$\alpha_1^{(nc)} = \alpha_2^{(nc)} = \frac{3}{4}$

$w_1 = 2$
$w_2 = 1$
$b_1 = 1$
$b_2 = 2$
Hypothesis: packets select their routes of travel from an origin to a destination so as to minimize their own travel cost.

3 possible routes
- cost of links are proportional to the fraction of users $x$ passing through it.

Difference with the previous example?
Hypothesis: packets select their routes of travel from an origin to a destination so as to minimize their own travel cost.

- The number of users is infinite
- Each of them has a negligible impact

Belongs to the class of “population games”
Definition: **Population game.**
- $Q$ non atomic populations, each of them of mass $\hat{d}_q$
- A finite set of strategies for each population
- A strategy distribution $y = (y_1, ..., y_Q)$, where $y_q$ is a vector containing the masses of the subset of population $q$ adopting each possible strategy
- The marginal payoff per unit of class $i$ of population $q$: $F_q^i(y)$

Definition: **Wardrop equilibrium.**
A state $\hat{y}$ is a Wardrop equilibrium if, for any population:
- All strategies being used by members of the population yield the same marginal payoff: $\forall i, j, y_q^i \neq 0, y_q^j \neq 0, F_q^i(\hat{y}) = F_q^j(\hat{y})$
- The marginal payoff associated to all strategies actually used by members is lower than it would be with any of the strategies not chosen.
Wardrop equilibria: application

1 population \((Q = 1)\), 3 possible strategies

Strategy distribution \(y = (y_1)\) with \(y_1 = (m_1, m_2, m_3)\)

Marginal payoff per unit:

\[
F_1^1(y) = 10 \times (m_1 + m_3) + (m_1 + 50) \\
= 11.m_1 + 10.m_3 + 50
\]

\[
F_1^2(y) = (m_2 + 50) + 10 \times (m_2 + m_3) \\
= 11.m_2 + 10.m_3 + 50
\]

\[
F_1^3(y) = 10 \times (m_1 + m_3) + (m_2 + 10) + 10 \times (m_2 + m_3) \\
= 10.m_1 + 20.m_3 + 11.m_2 + 10
\]
Wardrop equilibria: application

Total population mass
\[ m_1 + m_2 + m_3 = 6 \]
and:
\[
\begin{align*}
F_1^1(y) &= 11.m_1 + 10.m_3 + 50 \\
F_1^2(y) &= 11.m_2 + 10.m_3 + 50 \\
F_1^3(y) &= 10.m_1 + 11.m_2 + 20.m_3 + 10
\end{align*}
\]

Let \( \hat{y} \) be the strategy distribution at the Wardrop equilibria.

Then,
\[
\forall i, j, m_i \neq 0, m_j \neq 0, F_i^i(y) = F_j^j(y),
\]

and
\[
\forall i, j, m_i \neq 0, m_j = 0, F_i^i(y) < F_j^j(y).
\]
Wardrop equilibria: application

Total population mass
\[ m_1 + m_2 + m_3 = 6 \]

and:
\[ F_1^1(y) = 11.m_1 + 10.m_3 + 50 \]
\[ F_1^2(y) = 11.m_2 + 10.m_3 + 50 \]
\[ F_1^3(y) = 10.m_1 + 11.m_2 + 20.m_3 + 10 \]

Suppose only routes “a” and “b” are used \((m_3 = 0)\), then

\[ m_1 = 3 \quad \text{and} \quad F_1^1(y) = F_1^2(y) = 83. \]

But the single cost of a packet going through path “c” would be

\[ 10.m_1 + 11.m_2 + 10 = 73 < F_1^1(y), \]

hence \((m_1.m_2 \neq 0) \Rightarrow m_3 \neq 0.\)
Total population mass

\[ m_1 + m_2 + m_3 = 6 \quad \text{and:} \]

\[
F_1^1(y) = 11.m_1 + 10.m_3 + 50 \\
F_1^2(y) = 11.m_2 + 10.m_3 + 50 \\
F_1^3(y) = 10.m_1 + 11.m_2 + 20.m_3 + 10
\]

With similar arguments, we can show that \( m_1 . m_2 . m_3 \neq 0 \).

Hence \( F_1^1(y) = F_1^2(y) = F_1^3(y) \).

Then \( m_1 = m_2 \)

and \( 11.m_1 + 10.m_3 + 50 = 21.m_1 + 20.m_3 + 10 \),

hence \( 40 = 10.m_3 + 10.m_1 \).

Finally \( m_1 = m_2 = m_3 = 2 \) and \( F_1^1(\hat{y}) = F_1^2(\hat{y}) = F_1^3(\hat{y}) = 92 \).
Wardrop equilibria: application

\[ a \quad \begin{array}{c} 10.x \\ x + 50 \end{array} \quad b \quad \begin{array}{c} x + 50 \\ 10.x \end{array} \quad a \]

There is actually a simpler way :)

Potential games

Here, potential function
\[ \Phi(m_1, m_2, m_3) = \sum_{l \text{ links}} \int_{0}^{\alpha_l} c_l(u) du \]

with \( \alpha_l = \sum_{p \text{ paths}} m_l \delta_{l,p} \),
\[ \delta_{l,p} = \begin{cases} 1 & \text{if flow } l \text{ goes through path } p \\ 0 & \text{otherwise} \end{cases} \]

and \( c_l \) the cost of crossing link \( l \).

Then the Wardrop equilibria is the solution of:
\[ \hat{m} = (\hat{m}_1, \hat{m}_2, \hat{m}_3), \arg\min \Phi(m) \text{ subject to } \sum m_i = 6. \]
Wardrop equilibria: application

We saw that $F_1^1(\hat{y}) = F_1^2(\hat{y}) = F_1^3(\hat{y}) = 92$.

But also that, if only routes “a” and “b” were used ($m_3 = 0$), then

$$m_1 = 3 \text{ and } F_1^1(y) = F_1^2(y) = 83.$$  

(But the cost of a single packet going through path “c” would be $73 < F_1^1(y)$).
1 **Non-cooperative optimization**
- Nash Equilibria
- Braess Paradoxes
- Dynamic games
- Other equilibria

2 **Cooperative Games**
- Definitions of fairness
- Examples

3 **Conclusion**
- Other yet interesting topics...
- Cooperation versus Selfishness: A last example
Braess Paradoxes: definition

Context: urban transportation networks.
Hypothesis: travelers select their routes of travel from an origin to a destination so as to minimize their own travel cost or travel time.

Rate: 6

With 2 roads,
\[\text{Cost}_a = \text{Cost}_b = 83\]
Context: urban transportation networks.
Hypothesis: travelers select their routes of travel from an origin to a destination so as to minimize their own travel cost or travel time.

Rate: 6

With 2 roads,
\[ \text{Cost}_a = \text{Cost}_b = 83 \]

With 3 roads,
\[ \text{Cost}_a = \text{Cost}_b = \text{Cost}_c = 92 \]
Pareto-inefficient equilibria can exhibit unexpected behavior.

**Definition: Braess Paradox.**

There is a Braess Paradox if there exists two systems *ini* and *aug* such that

\[
ini < aug \quad \text{and} \quad \alpha^{(nc)}(ini) > \alpha^{(nc)}(aug).
\]

i.e. adding resources to the system may reduce the performances of **ALL** players simultaneously.
From the New York Times, Dec 25, 1990, Page 38, What if They Closed 42d Street and Nobody Noticed?, By GINA KOLATA:

“ON Earth Day this year, New York City’s Transportation Commissioner decided to close 42d Street, which as every New Yorker knows is always congested. ”Many predicted it would be doomsday,” said the Commissioner, Lucius J. Riccio. ”You didn’t need to be a rocket scientist or have a sophisticated computer queuing model to see that this could have been a major problem.” But to everyone’s surprise, Earth Day generated no historic traffic jam. Traffic flow actually improved when 42d Street was closed. “
Braess Paradoxes: applications
Non-cooperative scheduling with 1-port hypothesis

Hypothesis: the master can only send to 1 slave at a time.

Example
maître: \( W = 2.55 \)
3 machines: \((B_i, W_i) = (4.12, 0.41), (4.61, 1.31), (3.23, 4.76)\)
2 applications: \( b^1 = 1, w^1 = 2, b^2 = 2, w^2 = 1 \)

Equilibrium (ini): \( a^1 = 0.173, a^2 = 0.0365 \)
Equilibrium (\( W_2 = 5.4 \)): \( a^1 = 0.127, a^2 = 0.0168 \)
1 Non-cooperative optimization
   - Nash Equilibria
   - Braess Paradoxes
   - Dynamic games
   - Other equilibria

2 Cooperative Games
   - Definitions of fairness
   - Examples

3 Conclusion
   - Other yet interesting topics...
   - Cooperation versus Selfishness: A last example
Evolutionary games

- User strategies change with time as they adapt to the state
- Different possible dynamics:
  - Replicator dynamics:

  \[
  \dot{y}_q^s = y_q^s \left( F_q^s(y) - \frac{1}{\hat{d}_q} \sum_{i=1}^{S_q} y_q^i F_q^i(y) \right) .
  \]

  - Brown von Neumann Nash Dynamics (BNN):

  \[
  \gamma_q^s = \max \left\{ F_q^s(y) - \frac{1}{\hat{d}_q} \sum_{i=1}^{S_q} y_q^i F_q^i(y), 0 \right\} \quad \text{(excess payoff)}
  \]

  \[
  \dot{y}_q^s = \hat{d}_q \gamma_q^s - y_q^s \sum_{j=1}^{S_q} \gamma_j^s .
  \]

  (increase proportionally to the excess payoff / decrease proportionally to the sum of excess payoffs)
Evolutionary games

- Equilibria are called ESS (Evolutionary Stable Strategies) or
- Subset of Nash equilibria
- Stable by a deviation of a (small) fraction of users

Examples of applications:
- Power choice in ALOHA systems
  - Users can choose to transmit at high or low power (each packet)
  - High power has better chances of not being jammed
  - Low power save battery consumption
- Associations in wireless systems
1. Non-cooperative optimization
   - Nash Equilibria
   - Braess Paradoxes
   - Dynamic games
   - Other equilibria

2. Cooperative Games
   - Definitions of fairness
   - Examples

3. Conclusion
   - Other yet interesting topics...
   - Cooperation versus Selfishness: A last example
Many other frameworks of games:

- **Stackelberg equilibria**: strategic game between 2 players: a leader and a follower (used in pricing mechanisms of e-services). Over types used in pricing Bertrand competition, Cournot competition.

- **Stochastic games**: a type of dynamic games (i.e. evolving over time) where the transitions are stochastic - the next state is determined by a probability distribution depending on the current state and the chosen actions (Markov Decision Processes) (used to choose efficient scheduling strategies)
How to improve non-cooperative performance?

No universal solution, but several options:

**Correlated equilibria:**
- A correlator gives advice to each player
- (such that) the optimal strategy for each player is to follow the advice
- Nash equilibria $\subseteq$ Correlated equilibria

Interestingly, studies have shown that in certain cases, the correlator does not need to have any information on the system.

**Pricing mechanisms:**
- An entity gives money (reward) to players
- Each player strives to maximize its profit

Problem well studied in TCP-like networks (based on Lagrangian optimization)
Outline

1. Non-cooperative optimization
   - Nash Equilibria
   - Braess Paradoxes
   - Dynamic games
   - Other equilibria

2. Cooperative Games
   - Definitions of fairness
   - Examples

3. Conclusion
   - Other yet interesting topics...
   - Cooperation versus Selfishness: A last example
Axiomatic definition VS optimization problem

1. Pareto optimality
2. Symmetry
3. Invariance towards linear transformations

\[ \text{Independant to irrelevant alternatives} \]
\[ \text{Nash (NBS) / proportional fairness} \]
\[ \prod u_i \]

\[ \text{Monotonicity} \]
\[ \text{Raiffa-Kalai-Smorodinsky / max-min} \]
\[ \text{Recursively max}\{u_i|\forall j, u_i \leq u_j\} \]

\[ \text{Inverse monotonicity} \]
\[ \text{Thomson / Social welfare} \]
\[ \text{max} \sum u_i \]
Fairness: what does it amount to?

3 connections / 2 links.

Capacity constraints:

\[
\begin{align*}
    x_1 + x_0 &\leq 1, \\
    x_2 + x_0 &\leq 1, \\
    x_3 + x_0 &\leq 1.
\end{align*}
\]

4 unknowns and 3 (in)equations.

How to choose \(x_0\) among the Pareto optimal points?
Fairness: what does it amount to?

How to choose $x_0$ among the Pareto optimal points?

\[
\begin{align*}
    x_0 &= 0.5, \\
    x_1 &= x_2 = x_3 = 0.5 \\
    x_0 &= 0, \\
    x_1 &= x_2 = x_3 = 1 \\
    x_0 &= 0.25, \\
    x_1 &= x_2 = x_3 = 0.75
\end{align*}
\]

Max-Min fairness

Social Optimum

Proportional fairness

\[
\begin{align*}
    x_1 + x_0 &\leq 1, \\
    x_2 + x_0 &\leq 1, \\
    x_3 + x_0 &\leq 1.
\end{align*}
\]
Fairness family

Introduced by Mo and Walrand

utility

\[
\max \sum_{n \in N} \frac{f_n(x_n)^{1-\alpha}}{1-\alpha}
\]

allocation

player

Global Optimization

Proportional Fairness

Max-min Fairness

TCP Vegas

ATM (ABR)
Outline

1. Non-cooperative optimization
   - Nash Equilibria
   - Braess Paradoxes
   - Dynamic games
   - Other equilibria

2. Cooperative Games
   - Definitions of fairness
   - Examples

3. Conclusion
   - Other yet interesting topics...
   - Cooperation versus Selfishness: A last example
Fairness family: example
The COST network (Prop. fairness)
Fairness family: example
The COST network (Prop. fairness)

Corinne Touati (INRIA)
Introduction to Game Theory
Cooperative Games 26 / 33
Other examples

- **Bandwidth allocation**: in TCP-like networks, in UMTS-like networks (joint bandwidth / power)
- **Carriers allocation**: between operators in satellite systems

**Challenges:**

- **Non-convex systems**: wireless communications, load balancing systems
- **Developing distributed algorithms**
Outline

1 Non-cooperative optimization
   - Nash Equilibria
   - Braess Paradoxes
   - Dynamic games
   - Other equilibria

2 Cooperative Games
   - Definitions of fairness
   - Examples

3 Conclusion
   - Other yet interesting topics...
   - Cooperation versus Selfishness: A last example
Other hot topics in game theory

- **Mechanism design**: how to design rules of a game so as to achieve a specific outcome, even though each player is selfish. Done by setting up a structure in which each player has incentive to behave as the designer intends. (Leonid Hurwicz, Eric Maskin et Roger Myerson, Nobel 2007)

- **Auctions**: resource allocation in P2P, frequency allocation in wireless.


- **Fair division or cake cutting problem**: how to divide resource such that all recipients believe that they have received their fair share (envy-free). (Steven Brams, Alan Taylor)
Even more topics in game theory

- **Election**: Plurality (traditional) voting systems are not necessarily fair.

- **Stable marriages**: Problem of finding a matching, where no element of the first set prefers an element of the other set that also prefers the first element. “Stable Marriage and its Relation to Other Combinatorial Problems: An Introduction to the Mathematical Analysis of Algorithms”, Donald Knuth

- **Super-modular games**: utility functions are such that higher choices by one player make one’s own strategy higher look relatively more desirable.

- **Games with incomplete information or Bayesian games**: some player have private information about something relevant to their decision making

- **Games with imperfect information**: players do not perfectly observe the actions of other players.
**Outline**

1. Non-cooperative optimization
   - Nash Equilibria
   - Braess Paradoxes
   - Dynamic games
   - Other equilibria

2. Cooperative Games
   - Definitions of fairness
   - Examples

3. Conclusion
   - Other yet interesting topics...
   - Cooperation versus Selfishness: A last example
Example of enforced collaboration (set of rules enforced by the police)
While the purely non-cooperative approach would give...
Slides available at:
http://www-id.imag.fr/~touati/Talks/