Continuum Equilibrium

for

Massively Dense Ad-hoc Networks

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OBJECTIVES

- Background
- Models of costs
- Existing models and physics-inspired paradigms
- Continuous limits of a graph - road traffic perspective
- Optimization frameworks
- Numerical example

Massively Dense Limit

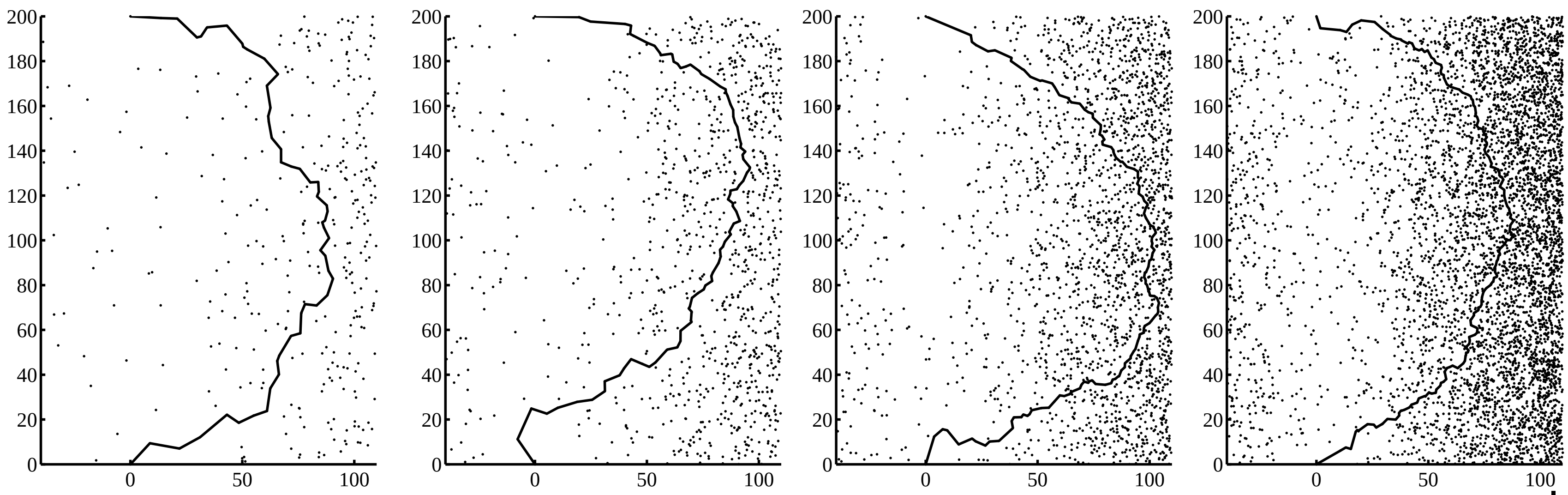


Figure 1: Minimum cost routes in increasingly large networks.

COST MODELS

Costs derived from capacity scaling

•Many models show how transport capacity scales with the number of nodes n or with their density λ.

•Assume routing with transport capacity of the order of $f(\lambda)$ at some region where the density of nodes is λ.

•**Capacity induced cost:** the density of nodes at a neighborhood of $\mathbf{x}$ required to carry a given flow.

•Assuming that a flow density $\mathbf{T}(\mathbf{x})$ is assigned at a neighborhood of $(\mathbf{x})$, the cost is taken to be

$$c(\mathbf{x}, \mathbf{T}(\mathbf{x})) = f^{-1}(\mathbf{T}(\mathbf{x})). \tag{1}$$

COST MODELS

•CSMA scheme with a fixed carrier sense range: $O\left(\frac{\sqrt{\lambda}}{\sqrt{\log(\lambda)}}\right)$

•Gupta and Kumar: capacity for optimally located nodes: $\Omega(\sqrt{\lambda})$.
Randomly located: $\Omega(\frac{\sqrt{\lambda}}{\sqrt{\log \lambda}})$.

•Using percolation theory Franceschetti et al close the gap and obtain $\Omega(\sqrt{\lambda})$

•Baccelli, Blaszczyszyn and Mühlethaler introduce introduce MSR (Multi-hop Spatial Reuse Aloha); attains also $O(\sqrt{\lambda})$.

•Tse and Glosglauser obtain $O(\lambda)$. Not massively dense ad-hoc networks

Conclusion: Both models of Gupta and Kumar as well as MSR yeild a quadratic cost

$$c(\mathbf{T}(\mathbf{x})) = k|\mathbf{T}(\mathbf{x})|^2$$

Congestion independent routing

•Standard metric: number of hops. Could be proportional to delay for negligible queueing delay. It depends only of the transmission range.

•If the range were constant then the cost density $c(\mathbf{x})$ is constant so that the cost of a path is its length in meters.

•Assume the range $R(\mathbf{x})$ depends on local weather conditions at $\mathbf{x}$ Assume that the range is scaled to go to 0 as the total density λ grows

$$r(\mathbf{x}) := \lim_{\lambda\to\infty} \frac{R(\mathbf{x})}{f(\lambda)} \text{ where } \lim_{\lambda\to\infty} f(\lambda) = 0.$$

•Then in the dense limit, the density of nodes that participate in forwarding packets along a path is $1/r(\mathbf{x})$ and the path cost is the integral of this density along the path.

•The influence of varying radio conditions on the range can be eliminated using power control that can equalize the hop distance.

Costs related to energy

•In the absence of capacity constraints, the cost can represent energy consumption.

•Useful even for multi-hop ad-hoc within a single cell of 802.11 IEEE wireless LAN.

•The cost can take into account the scaling of the nodes obtained when there are energy constraints.

•Example, assuming random deployment of nodes, where each nodes has traffic to send to another randomly selected node, the capacity (in bits per Joule):

$$f(\lambda) = \Omega\left((\lambda/\log\lambda)^{(q-1)/2}\right)$$

where q is the path-loss [Rodoply and Meng, 2007]

Preliminary

•Distributed traffic sources and sinks $\rho(\mathbf{x})$.

•Flow conservation:

$$\int\limits_{\Phi_0} \rho(\mathbf{x}) d\mathbf{x} = \oint\limits_{\partial\Phi_0} [\mathbf{T}(\ell) \cdot \mathbf{n}(\ell)] d\ell \tag{2}$$

Follows from Green's Theorem:

$$\nabla \cdot \mathbf{T}(\mathbf{x}) := \frac{\partial T_1(\mathbf{x})}{\partial x_1} + \frac{\partial T_2(\mathbf{x})}{\partial x_2} = \rho(\mathbf{x}), \tag{3}$$

where "$\nabla\cdot$" is the divergence operator.

•**Multiclass:** ν traffic clases.

•Define for each $\mathbf{T}^j$ and ρ^j. $\mathbf{T}(\mathbf{x})$:= total flow at $\mathbf{x}$.

•Global optimization: minimize Z over $\mathbf{T}(\cdot)$ where

$$Z = \min_{\{T_i^j\}} \int_{\Phi} g(\mathbf{x}, \mathbf{T}(\mathbf{x})) \mathrm{d}x_1 \mathrm{d}x_2 \text{ subject to } \nabla \cdot \mathbf{T}^j(\mathbf{x}) = \rho^j(\mathbf{x}),\ j = 1, ..., \nu\ \forall \mathbf{x} \in \Phi. \tag{4}$$

where $\mathbf{T}(\cdot)$ is in $(H^1(\Phi)^{2\nu}$ - the 2ν scalar components $H^1(\Phi)$, the Sobolev space of L^2 functions with L^2 first partial derivatives.

Directional Antennas

•For energy efficiency, directional antennas allow transmission from North to South or from West to East.

•Thus $T_1^j \geq 0,\ T_2^j \geq 0$, $j = 1, ..., \nu$.
In the dense limit, a curved path can be viewed as a limit of a path with many such hops

•Extend framework by Dafermos [1980]

Assumptions on the cost

•**Individual cost:** Vector valued $\mathbf{g} = \mathbf{g}(\mathbf{x}, \mathbf{T}(\mathbf{x}))$:
We allow the cost g_1 for a horizontal (West-East) transmission to be different than the cost g_2 for a vertical transmission (North-South).

•The local cost corresponding to the global optimization problem is given by $g(\mathbf{x}, \mathbf{T}(\mathbf{x})) = \mathbf{g}(\mathbf{x}, \mathbf{T}(\mathbf{x})) \cdot \mathbf{T}(\mathbf{x})$ if it is perceived as the sum of costs of individuals.

•Or it is the integral of a cost density where the latter has the form $g = \widehat{g}(\mathbf{x}, \mathbf{T}(\mathbf{x}))$.

•The local cost $g(\mathbf{x}, \mathbf{T}(\mathbf{x}))$ is non-negative, convex increasing in both arguments.

The **boundary conditions** will be determined by the options that travelers have in selecting their origins and/or destinations. Examples of the boundary conditions are:

•Assignment problem: predetermined origins and destinations

•Combined distribution and assignment: users have predetermined origins and are free to choose their destinations

•Combined generation, distributions and assignment: users are free to choose their origins, their destinations, as well as their travel paths.

•The problem formulation is again to minimize Z as defined in (4).

Kuhn-Tucker conditions

•Define the Lagrangian,

$$L^\zeta(\mathbf{x},\mathbf{T}) := \int_\Phi \ell^\zeta(\mathbf{x},\mathbf{T})\mathrm{d}\mathbf{x} \text{ where } \ell^\zeta(\mathbf{x},\mathbf{T}) = g(\mathbf{x},\mathbf{T}(\mathbf{x})) - \sum_{j=1}^{\nu} \zeta^j(\mathbf{x})\Big[\nabla\cdot\mathbf{T}^j(\mathbf{x}) - \rho^j(\mathbf{x})\Big]$$

where $\zeta = \{\zeta^j(\mathbf{x})\}$ are Lagrange multipliers.

•g convex and the constraints are affine, so Kuhn-Tucker (KT) conditions hold: for any admissible variation $\delta\mathbf{T}$ from the optimal $\mathbf{T}$,

$$DL^\zeta \cdot \delta\mathbf{T} \geq 0.$$

therefore here

$$\int_\Omega \sum_j \langle \nabla_{\mathbf{T}^j} g(\mathbf{x},\mathbf{T}(\mathbf{x})), \delta\mathbf{T}^j(\mathbf{x})\rangle \,\mathrm{d}\mathbf{x} - \int_\Omega \sum_j \zeta^j(\mathbf{x})\nabla\cdot\delta\mathbf{T}^j(\mathbf{x})\,\mathrm{d}\mathbf{x} \geq 0.$$

Integrating by parts with Green's formula, this is equivalent to

$$\int_\Omega \sum_j \left[\langle \nabla_{\mathbf{T}^j} g, \delta\mathbf{T}^j\rangle + \langle \nabla_{\mathbf{x}}\zeta^j, \delta\mathbf{T}^j\rangle\right]\mathrm{d}\mathbf{x} - \int_{\partial\Omega} \sum_j \zeta^j \langle \delta\mathbf{T}^j, \mathbf{n}\rangle \,\mathrm{d}\ell \geq 0\,.$$

•This leads to the conditions for $i = 1, 2$:

$$\frac{\partial g(\mathbf{x}, \mathbf{T})}{\partial T_i^j(\mathbf{x})} + \frac{\partial \zeta^j(\mathbf{x})}{\partial x_i} = 0 \quad \text{if } T_i^j(\mathbf{x}) > 0, \tag{5}$$

$$\frac{\partial g(\mathbf{x}, \mathbf{T})}{\partial T_i^j(\mathbf{x})} + \frac{\partial \zeta^j(\mathbf{x})}{\partial x_i} \geq 0 \quad \text{if } T_i^j(\mathbf{x}) = 0. \tag{6}$$

•Special cases where $g(\mathbf{x}, \mathbf{T}(\mathbf{x})) = \sum_{i=1,2} g_i(\mathbf{x}, \mathbf{T}(\mathbf{x})) T_i(\mathbf{x})$,

1. Monomial cost per packet:

$$g_i(\mathbf{x}, \mathbf{T}(\mathbf{x})) = c_i(\mathbf{x})\Big(T_i(\mathbf{x})\Big)^{\beta} \tag{7}$$

for some $\beta > 1$. Then (5)-(6) simplify to

$$(\beta+1)c_i(\mathbf{x})\Big(T_i(\mathbf{x})\Big)^{\beta} + \frac{\partial \zeta^j(\mathbf{x})}{\partial x_i} = 0 \quad \text{if } T_i^j(\mathbf{x}) > 0, \tag{8}$$

$$(\beta+1)c_i(\mathbf{x})\Big(T_i(\mathbf{x})\Big)^{\beta} + \frac{\partial \zeta^j(\mathbf{x})}{\partial x_i} \geq 0 \quad \text{if } T_i^j(\mathbf{x}) = 0. \tag{9}$$

2. Affine cost per packet:

$$g_i(\mathbf{x}, \mathbf{T}(\mathbf{x})) = c_i(\mathbf{x})T_i(\mathbf{x}) + d_i(\mathbf{x}). \tag{10}$$

Then (5)-(6) simplify to

$$2c_i(\mathbf{x})T_i + d_i(\mathbf{x}) + \frac{\partial \zeta^j(\mathbf{x})}{\partial x_i} = 0 \quad \text{if } T_i^j(\mathbf{x}) > 0, \tag{11}$$

$$2c_i(\mathbf{x})T_i + d_i(\mathbf{x}) + \frac{\partial \zeta^j(\mathbf{x})}{\partial x_i} \geq 0 \quad \text{if } T_i^j(\mathbf{x}) = 0.$$

User optimization & Wardrop equilib

Congestion independent cost

•The local cost depends on the direction of the flow but not on its size.

•$c_1(\mathbf{x})$ for a flow that is locally horizontal, $c_2(\mathbf{x})$ for vertical.

•First assume that c_1 and c_2 do not depend on $\mathbf{T}$.

The cost for traveling along a path p is

$$\mathbf{c}_p = \int_p \mathbf{c} \cdot \mathbf{dx}. \tag{12}$$

•Let $V^j(\mathbf{x})$ be the minimum cost to go from a point $\mathbf{x}$ to a set B^j, $j = 1, ..., \nu$.

•Assume that $B^j \subset \partial\Phi$ is part of the boundary of Φ and that the south-east corner of the rectangle is included in all B^j's.

•Then

$$V^j(\mathbf{x}) = \min\Big(c_1(\mathbf{x})\mathrm{d}x_1 + V^j(x_1 + \mathrm{d}x_1, x_2), c_2(\mathbf{x})\mathrm{d}x_2 + V^j(x_1, x_2 + \mathrm{d}x_2)\Big) \quad (13)$$

•This can be written as

$$0 = \min\left(c_1(\mathbf{x}) + \frac{\partial V^j(\mathbf{x})}{\partial x_1}, c_2(\mathbf{x}) + \frac{\partial V^j(\mathbf{x})}{\partial x_2}\right) \quad (14)$$

•Is V^j differentiable? what if not?

•If V^j is differentiable then, under suitable conditions, it is the unique solution (14).

•In the case that V^j is not every-where differentiable then, under suitable conditions, it is the unique viscosity solution of (14).

•Numerical approaches for solving the HJB equation: discretization provides a discrete dynamic programming

Geometry of minimum cost paths

•Define the function

$$U(\mathbf{x}) = \frac{\partial c_2}{\partial x_1}(\mathbf{x}) - \frac{\partial c_1}{\partial x_2}(\mathbf{x}) \quad \forall \mathbf{x} \in \Omega.$$

•The structure of the minimum cost path depends on the costs through the sign of the function U.

•We assume U has the same sign everywhere (see Fig. 3),

•or there are two regions in R, one with $U > 0$ and one with $U < 0$, separated by a curve on which $U = 0$ (e.g. Fig. 4).

$$
\begin{aligned}
\Gamma_1^+ &= \{0 \le x_1 \le a, \quad x_2 = 0\} \\
\Gamma_2^+ &= \{x_1 = a, \quad 0 \le x_2 \le b\} \\
\Gamma_3^- &= \{0 \le x_1 \le a, \quad x_2 = b\} \\
\Gamma_4^- &= \{x_1 = 0, \quad 0 \le x_2 \le b\}.
\end{aligned}
$$

Figure 2: The boundaries of the region R.

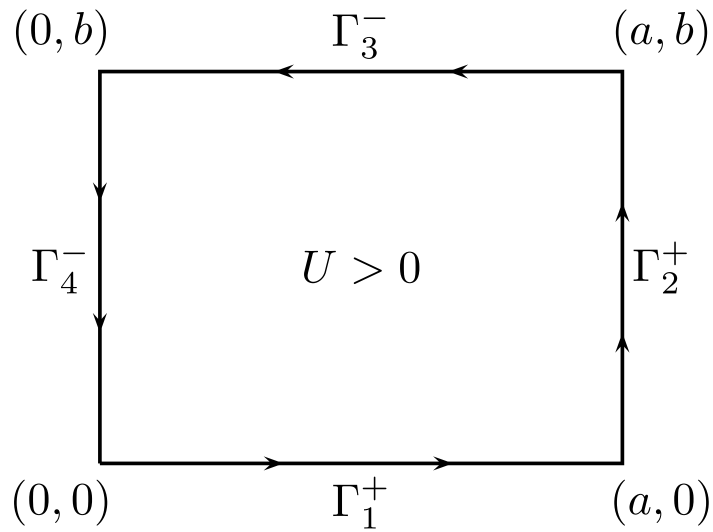


Figure 3: The region R. The case where $U > 0$.

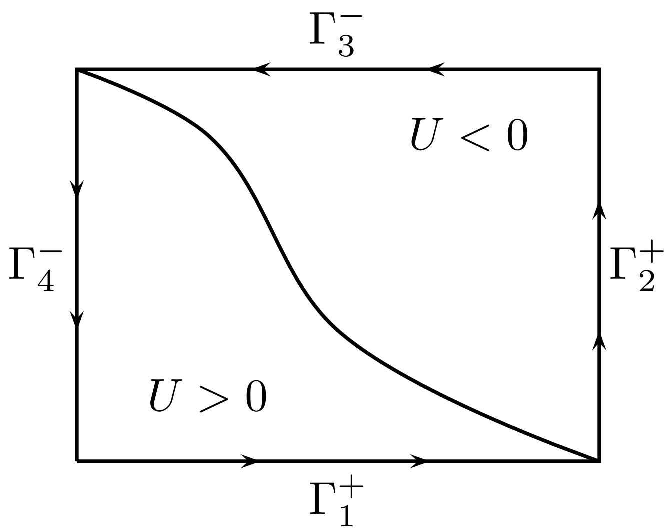


Figure 4: The case of two regions separated by a curve. Case **??**.

The function U has the same sign over the whole region R

•**Theorem:** (Point to point optimal path) Suppose that a point $\mathbf{x}^O = (x_1^O, x_2^O)$ in $\mathring{R}$ (the interior of R), wants to send a packet to a point $\mathbf{x}^D = (x_1^D, x_2^D)$ in $\mathring{R}$.

i. If $U > 0$ in the region $R_{OD} = \{(x_1, x_2)$ such that $x_1^O \le x_1 \le x_1^D, x_2^O \le x_2 \le x_2^D\}$, except perhaps from a set of Lebesgue measure zero, then there is an optimal path given by (see Fig. 5):

$$\gamma^{\text{opt}} = \gamma_H \cup \gamma_V \text{ where}$$

$$\gamma_H = \{(x_1, x_2) \text{ such that } x_1^O \le x_1 \le x_1^D, x_2 = x_2^O\}$$

$$\gamma_V = \{(x_1, x_2) \text{ such that } x_1 = x_1^D, x_2^O \le x_2 \le x_2^D\}.$$

ii. If $U < 0$ in that region except perhaps from a set of Lebesgue measure zero, then there is an optimal path given by (see Fig. 6):

$$\gamma^{\text{opt}} = \gamma_V \cup \gamma_H \text{ where}$$

$$\gamma_V = \{(x_1, x_2) \text{ such that } x_1 = x_1^O, x_2^O \leq x_2 \leq x_2^D\}$$
$$\gamma_H = \{(x_1, x_2) \text{ such that } x_1^O \leq x_1 \leq x_1^D, x_2 = x_2^D\}.$$

iii. In both cases, γ^{opt} is unique up to a zero Lebesgue measure. (i.e. the Lebesgue measure of the area between γ^{opt} and any other optimal path is zero).

Green's Theorem

Let $\Omega \subseteq \mathcal{X}$ be a region of the space, and let Γ be its boundary. Suppose that $P, Q \in \mathcal{C}^1(\Omega)$ (We denote $\mathcal{C}^1(\Omega)$ the set of functions that are differentiable and whose partial derivatives are continuous on Ω.) Then

$$\oint_{\Gamma^+} P dx + Q dy = \int_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy. \tag{15}$$

Proof.-

•valid path (each subpath can be decomposed of paths either from N to S or from W to E (or limits).

•Consider a valid path γ_C joining $\mathbf{x}^O$ to $\mathbf{x}^D$, and assume that the Lebesgue measure of the area between γ^{opt} and γ_C is nonzero.

•We call such path, the comparison path (see Fig. 5 for the case $U > 0$ and Fig. 6 for $U < 0$).

Denote
$\xi^- = \gamma_H^+ \cup \gamma_V^- \cup \gamma_C^+$
Let Ω_1 denote the bounded area described by ξ^-.

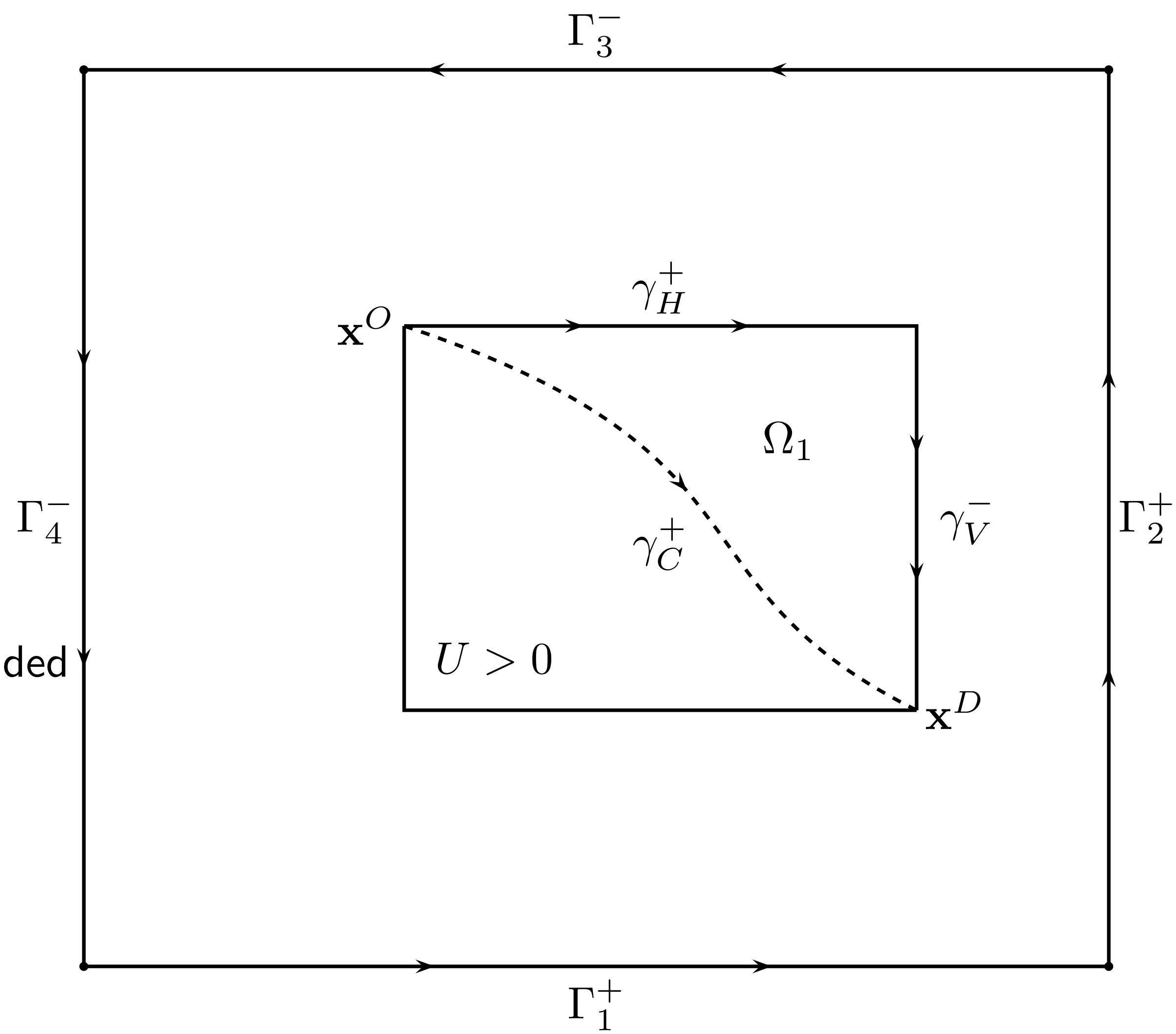


Figure 5: Optimal path for $U > 0$.

•(i) Showing that the cost over path γ^{opt} is optimal is equivalent to showing that the integral of the cost over the closed path ξ^- is negative

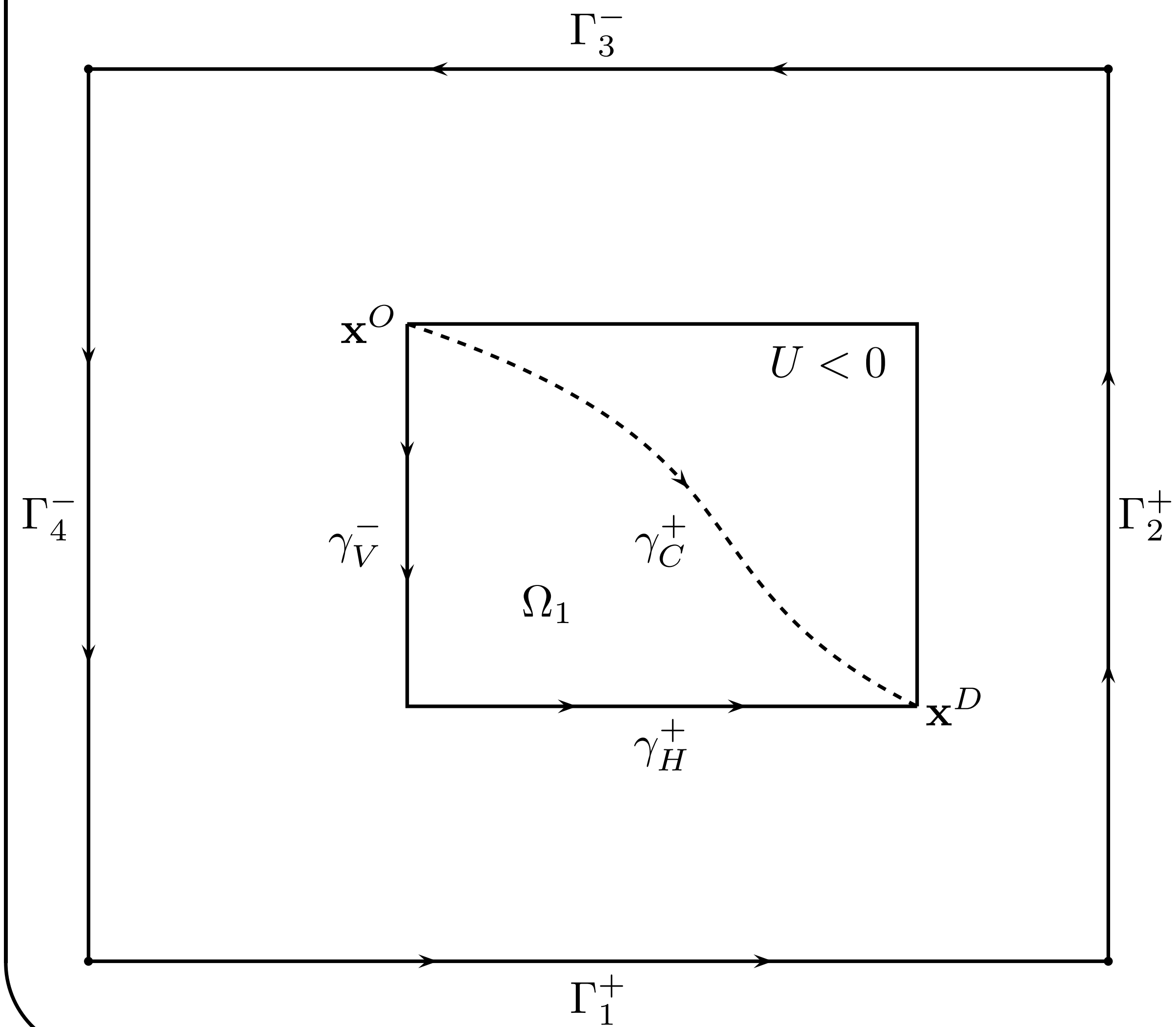
•Using Green Theorem we obtain

$$\oint_{\xi^-} \mathbf{c} \cdot \mathbf{dx} = - \int_{\Omega_1} U(\mathbf{x}) dS$$

which is strictly negative since $U > 0$ a.e. on R.

•Decomposing the left integral, this concludes the proof of (i), and establishes the corresponding statement on uniqueness in (iii).

•(ii) is obtained similarly. $\diamond$


Γ_3^-
$\mathbf{x}^O$
$U < 0$
Γ_4^-
γ_V^-
γ_C^+
Γ_2^+
Ω_1
$\mathbf{x}^D$
γ_H^+
Γ_1^+

Congestion dependent cost

•We now add to c_1 the dependence on T_1 and to c_2 the dependence on T_2.

•Let $V^j(\mathbf{x})$ be the minimum cost to go from a point $\mathbf{x}$ to B^j at equilibrium. (13) still holds but this time with c_i that depends on T_i^j, $i = 1, 2.$ and on the total flows T_i, $i = 1, 2.$

•Thus (14) becomes

$$0 = \min_{i=1,2} \left(c_i(\mathbf{x}, T_i) + \frac{\partial V^j(\mathbf{x})}{\partial x_i} \right). \tag{16}$$

•We note that if $T_i^j(\mathbf{x}) > 0$ then by the definition of the equilibrium, i attains the minimum at (16). Hence (16) implies the following relations for $i = 1, 2$:

$$c_i(\mathbf{x}, T_i) + \frac{\partial V^j}{\partial x_i} = 0 \quad \text{if} \quad T_i^j > 0, \tag{17}$$

$$c_i(\mathbf{x}, T_i) + \frac{\partial V^j}{\partial x_i} \geq 0 \quad \text{if} \quad T_i^j = 0. \tag{18}$$

Beckmann type transformation

•Consider single class

•(17)-(18) are the same as the KT conditions (5)-(6) except that $c_i(\mathbf{x}, T_i)$ in the former are replaced by $\partial g(\mathbf{x}, \mathbf{T})/\partial T_i(\mathbf{x})$ in the latter.

•We conclude that if there exists a scalar function (potential) $\psi(\mathbf{x}, \mathbf{T})$ such that for both $i = 1, 2$:

$$c_i(\mathbf{x}, T_i) = \frac{\partial \psi(\mathbf{x}, \mathbf{T})}{\partial T_i(\mathbf{x})}$$

then the user equilibrium flow is the one obtained from the global optimization problem where we use $\psi(\mathbf{x}, \mathbf{T})$ as local cost.

•Hence, the Wardrop equilibrium is obtained as the solution of

$$\min_{T(\mathbf{x}),\ x \in \Phi} \int_\Phi \psi(\mathbf{x}, \mathbf{T}) d\mathbf{x} \qquad \text{subject to } \nabla \cdot \mathbf{T}(\mathbf{x}) = \rho(\mathbf{x}), \qquad \forall (\mathbf{x}) \in \Phi.$$

•The potential ψ is given by

$$\psi(\mathbf{x}, \mathbf{T}) = \sum_{i=1,2} \int_0^{T_i(\mathbf{x})} c_i(\mathbf{x}, s)\mathrm{d}s$$

•Assume costs are given as a power of the flow as defined in eq (7), we observe that equations (17)-(18) coincide with equations (8)-(9) (up-to a multiplicative constant of the cost).

•We conclude that for such costs, the user equilibrium and the global optimization solution coincide.

Competitive routing

•Routing decisions taken by a finite number N of competing service providers.

•There are N classes, one per player. T^j is the flow distribution of class j and ρ^j corresponds to the distribution of the external sources and/or sinks.

•**Assumption A1:** The local cost $g^j(\mathbf{x}, \mathbf{T}^j, \mathbf{T})$ corresponding to player j may depend on

- the total horizontal and the total vertical flow and not directly on the amount of flow of each class.
- the total horizontal and vertical flow of that same player,
- The location.

•Player j (controlling the routing of class j) minimizes the total cost Z^j for its traffic, where

$$Z^j = \int_\Phi g^j(\mathbf{x}, \mathbf{T}) d\mathbf{x},$$

subject to

$$\nabla \cdot \mathbf{T}^j(\mathbf{x}) = \rho^j(\mathbf{x}), \qquad \forall \mathbf{x} \in \Phi.$$

•Define the Lagrangian for player j as

$$L^{\zeta,j}(\mathbf{x},\mathbf{T}) := \int_\Phi \ell^{\zeta,j}(\mathbf{x},\mathbf{T})\mathrm{d}\mathbf{x} \textbf{ where } \ell^{\zeta,j}(\mathbf{x},\mathbf{T}) = g^j(\mathbf{x},\mathbf{T}(\mathbf{x})) + \zeta^j(\mathbf{x})\Big[\nabla\cdot\mathbf{T}^j(\mathbf{x}) - \rho^j(\mathbf{x})\Big]$$

•The KT conditions corresponding to this problem are for $i = 1, 2$

$$\frac{\partial \ell^{\zeta,j}(\mathbf{x},\mathbf{T})}{\partial T_i^j(\mathbf{x})} = 0 \quad \text{if } T_i^j(\mathbf{x}) > 0,$$

$$\frac{\partial \ell^{\zeta,j}(\mathbf{x},\mathbf{T})}{\partial T_i^j(\mathbf{x})} \geq 0 \quad \text{if } T_i^j(\mathbf{x}) = 0$$

•We thus obtain for $i = 1, 2$:

$$\frac{\partial g^j(\mathbf{x},\mathbf{T})}{\partial T_i^j(\mathbf{x})} + \frac{\partial \zeta^j(\mathbf{x})}{\partial x_i} = 0 \quad \text{if } T_i^j(\mathbf{x}) > 0, \tag{19}$$

$$\frac{\partial g^j(\mathbf{x},\mathbf{T})}{\partial T_i^j(\mathbf{x})} + \frac{\partial \zeta^j(\mathbf{x})}{\partial x_i} \geq 0 \quad \text{if } T_i^j(\mathbf{x}) = 0. \tag{20}$$

•Assume that the per packet local *cost density* is linear in the congestion:

$$g^j(\mathbf{x}, \mathbf{T}(\mathbf{x})) = \sum_{i=1,2} \Big(c_i(\mathbf{x}) T_i(\mathbf{x}) + d_i(\mathbf{x})\Big) T_i^j(\mathbf{x}) \tag{21}$$

•Then

$$\frac{\partial g^j(\mathbf{x}, \mathbf{T})}{\partial T_i(\mathbf{x})} = c_i(\mathbf{x})(T_i(\mathbf{x}) + T_i^j(\mathbf{x})) + d_i(\mathbf{x}).$$

•Eq. (19)-(20) simplify to

$$c_i(\mathbf{x})(T_i(\mathbf{x}) + T_i^j(\mathbf{x})) + d_i(\mathbf{x}) + \frac{\partial \zeta^j(\mathbf{x})}{\partial x_i} = 0 \quad \text{if } T_i^j(\mathbf{x}) > 0, \tag{22}$$

$$c_i(\mathbf{x})(T_i(\mathbf{x}) + T_i^j(\mathbf{x})) + d_i(\mathbf{x}) + \frac{\partial \zeta^j(\mathbf{x})}{\partial x_i} \geq 0 \quad \text{if } T_i^j(\mathbf{x}) = 0.$$

•Assume now that at equilibrium, there is positive density of flow $\mathbf{T}^j(\mathbf{x})$ over the whole plane Φ for every player j. Then (22) holds for all j. Summing over j we obtain for $i = 1, 2$:

$$(n+1) T_i(\mathbf{x}) c_i(\mathbf{x}) + n d_i(\mathbf{x}) + \frac{\partial \zeta(\mathbf{x})}{\partial x_i} \geq 0$$

where $\zeta = \sum_{j=1}^{n} \zeta^j$.

•We thus obtained the KT conditions for the globally optimal problem in which $\rho = \sum_j \rho^j$ and in which the local cost is given by

$$g(\mathbf{x}, \mathbf{T}(\mathbf{x})) = \sum_{i=1,2} \Big(\frac{n+1}{2} c_i(\mathbf{x}) T_i(\mathbf{x}) + n d_i(\mathbf{x}) \Big) T_i(\mathbf{x}) \tag{23}$$

Hence as in the Wardrop case, it is possible to transform the game into an equivalent globally optimal problem with a single decision maker.