On the performance of congestion games for optimum satisfiability problems

Laurent Gourvès

CNRS LAMSADE - Université Paris Dauphine

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joint work with A. Giannakos, J. Monnot & V. Th. Paschos

Congestion games

$$\langle N, M, (A_i)_{i \in N}, (c_j)_{j \in M} \rangle$$

- $N = \{1, \ldots, n\}$, selfish players
- $M = \{1, \ldots, m\}$, resources
- $A_i \subseteq 2^M$, strategy space of player i
- c_j, cost function associated with resource j

Rosenthal's potential function $\Phi : A_1 \times A_2 \times \ldots \times A_n \to \mathbb{Z}$

Existence of pure Nash equilibria

Congestion games

Two fundamental questions:

1. Time to converge to a pure NE
2. Performance deterioration due to selfish behavior

Two devastating answers:

1. finding a pure NE is **PLS**-complete
   [Fabrikant, Papadimitriou & Talwar STOC 2004]
2. very far from the social optimum
   [Christodoulou & Koutsoupias STOC 2005]

Motivation

Previous results hold for the general (worst) case

> If the strategy space of each player consists of the bases of a matroid over the set of resources, then the length of all best response sequences are polynomially bounded in the number of players and resources [Ackermann, Röglin and Vöcking. FOCS 2006]

We need to explore the underlying combinatorial structure of congestion games:

the focus of this talk is on :

- Performance deterioration due to selfish behavior
- paradigmatic problem in CO: SAT

MAX SAT

- variable set: $X = \{x_1, \ldots, x_n\}$
- clause set: $\mathcal{C} = \{C_1, \ldots, C_m\}$
  - a clause is a disjunction of literals (e.g. $x_1 \vee \overline{x}_3$)
- each clause C_j has a weight w_j

MAX SAT: Find a truth assignment that maximizes the weight of satisfied clauses

MAX E $k-$SAT: each clause has exactly k literals

MAX $k-$SAT: each clause has at most k literals

An example

$X = \{x_1, x_2, x_3, x_4, x_5\}$

$\mathcal{C} = \{C_1, C_2, C_3, C_4\}$

C_1	$\overline{x}_1$			$w_1 = 4$
C_2	x_1	$\vee$	$\overline{x}_3$	$w_2 = 1$
C_3	x_2	$\vee$	$\overline{x}_4$	$w_3 = 2$
C_4	x_3	$\vee$	x_5	$w_4 = 3$

The truth assignment (*true*, *true*, *true*, *false*, *false*) has weight 6

A MAX SAT game

Each variable x_i is controlled by a selfish player with strategy space $\{true, false\}$

General payment scheme:

- a variable receives nothing from a clause she does not satisfy
- if ℓ variables satisfy C_j then each of them get $f(\ell)\, w_j$

utility u_i of player i = sum of rewards received over all clauses

MAX SAT game $\in$ congestion games

A "fair" payment scheme

$f(\ell) = 1/\ell \rightarrow$ if ℓ variables satisfy C_j then each of them get w_j/ℓ

C_1	$\overline{x}_1$			$w_1 = 1$
C_2	x_1	$\vee$	x_2	$w_2 = 1$
C_3	x_1	$\vee$	x_3	$w_3 = 1$

(*true*, *true*, *true*) is a Nash equilibrium for the fair payment scheme

(*false*, *true*, *true*) is optimal

What is the deterioration of the system's performance due to the lack of coordination between selfish agents?

Can we give a better payment scheme?

Price of Anarchy

Capturing the deterioration of the system's performance due to the lack of coordination between selfish agents

(pure) Price of Anarchy $\boxed{\min_{\vec{a} \in PNE} \frac{Q(\vec{a})}{Q(\vec{a}^*)}}$

Koutsoupias & Papadimitriou, 99

System's state: $\vec{a} \in A = \times_{i \in N} A_i$

Overall quality of the system $Q : A \rightarrow \mathbb{R}_{\geq 0}$

$PNE(G)$: set of (pure) Nash Equilibria of G

$\vec{a}^* = argmax_{\vec{a} \in A}\{Q(\vec{a})\}$

PoA

Theorems

- Under the fair payment scheme, the PoA of the MAX E $k-$SAT game is $\frac{k}{k+1}$
- Under the fair payment scheme, the PoA of the MAX $k-$SAT game is $\frac{k}{2k-1}$

Proof sketch

THM: The PoA of the MAX E $k-$SAT game is $\frac{k}{k+1}$

Assumption: $\forall i \quad a_i = true$

Nash property: $u_i(\vec{a}) \geq u_i\big((\vec{a}_{-i}, false)\big)$

$SAT =$ weight of the Nash equilibrium
$UN =$ weight of the clauses not satisfied by the NE

$$\begin{aligned} \sum_{i \in N} u_i(\vec{a}) &\geq \sum_{i \in N} u_i\big((\vec{a}_{-i}, false)\big) \\ SAT &\geq k\, UN \\ (k+1)SAT &\geq k\, SAT + k\, UN \geq k\, OPT \end{aligned}$$

Tightness

C_1	$\overline{x}_1$	$\vee$	$\overline{x}_2$	$\vee$	$\ldots$	$\vee$	$\overline{x}_k$	$w_1 = 1$
C_2	x_1	$\vee$	x_2	$\vee$	$\ldots$	$\vee$	x_k	$w_2 = k$

$(true, true, \ldots, true)$ is a NE satisfying C_2, $SAT = k$

$(false, true, true, \ldots, true)$ satisfies $\{C_1, C_2\}$, $OPT = k + 1$

Connection to local search

congestion game		local search
system's state	$\leftrightarrow$	feasible solution
potential function Φ	$\leftrightarrow$	cost function c
unilateral move	$\leftrightarrow$	neighborhood
Nash equilibrium	$\leftrightarrow$	local optimum
PoA	Overall quality Q	locality gap

$c = Q$ in *standard* local search

c not necessarily equal to Q in **non oblivious** local search

A better payment scheme for the MAX E 2−SAT game

We focus on the MAX E 2−SAT game

Non oblivious payment scheme:

- if 1 variable satisfies C_j then she gets w_j
- if 2 variables satisfy C_j then each one gets $w_j/3$

THM: The PoA of the MAX E 2−SAT game under the non oblivious payment scheme is $3/4$

Recall that PoA$= 2/3$ under the fair payment scheme

Proof sketch

$SAT =$ weight of the Nash equilibrium
$ONE =$ weight of the clauses satisfied by **one** variable
$TWO =$ weight of the clauses satisfied by **two** variables
$UN =$ weight of the clauses not satisfied

$$
\begin{aligned}
\sum_{i \in N} u_i(\vec{a}) &\geq \sum_{i \in N} u_i\big((\vec{a}_{-i}, false)\big) \\
ONE + \frac{2}{3} TWO &\geq 2UN + \frac{1}{3} ONE \\
\frac{2}{3} ONE + \frac{2}{3} TWO &\geq 2UN \\
\frac{2}{3} SAT &\geq 2UN \\
SAT &\geq 3UN \\
4\,SAT &\geq 3UN + 3SAT \geq 3\,OPT
\end{aligned}
$$

Tightness

$k = 2$

$$\begin{array}{ccc} \overline{x}_1 & \vee & \overline{x}_2 \\ \overline{x}_3 & \vee & \overline{x}_4 \\ x_1 & \vee & x_2 \\ x_2 & \vee & x_3 \\ x_3 & \vee & x_4 \\ x_4 & \vee & x_1 \end{array}$$

$(true, true, true, true, true, true)$ is a NE satisfying 4 clauses

$(false, true, false, true, true, true)$ satisfies 6 clauses

Better payment schemes

- There exists a payment scheme such that the PoA of the MAX E $k-$SAT game is $1 - \frac{1}{2^k}$
- There exists a payment scheme such that the PoA of the MAX $k-$SAT game is $\frac{2}{3}$

There exists a non oblivious local search algorithm with locality gap $1 - 1/2^k$ for MAX E k SAT [Khanna et al, 98]

There exists a non oblivious local search algorithm with locality gap $2/3$ for MAX k SAT

Best payment schemes

General payment scheme:

- a variable receives nothing from a clause she does not satisfy
- if ℓ variables satisfy C_j then each of them get $f(\ell)\, w_j$

There exists a family of instances of the MAX E $k-$ SAT game such that PoA$= 1 - 1/2^k$

$$\overline{x}_1 \vee \overline{x}_2$$
$$x_1 \vee \overline{x}_3$$
$$x_2 \vee \overline{x}_4$$
$$x_3 \vee x_4$$

Other results

How good is the best Nash equilibrium?

Price of Stability $\boxed{\max_{\vec{a} \in PNE} \frac{Q(\vec{a})}{Q(\vec{a}^*)}}$

Lemma : PoA = PoS for the max sat game

A MIN SAT game

MIN SAT: Find a truth assignment that minimizes the weight of satisfied clauses

Players are penalized for satisfying a clause

Fair penalty scheme:

- a variable pays nothing for a clause she does not satisfy
- if ℓ variables satisfy C_j then each one must **pay** w_j/ℓ

Theorems:

The PoA of the MIN $k-$ SAT game is k

The PoS of the MIN $k-$ SAT game is $1 + \frac{1}{2} + \ldots + \frac{1}{k} = H(k)$

PoA

C_0	$\overline{x_1} \vee \overline{x_2} \vee \cdots \vee \overline{x_k}$	$w_0 = 1$
C_1	x_1	$w_1 = 1$
C_2	x_2	$w_2 = 1$
$\vdots$	$\vdots$	$\vdots$
C_k	x_k	$w_k = 1$

PoS

Rosenthal's potential function of the MIN $k-$ SAT game

$$\Phi(a) = \sum_{j=1}^{k} H(j)\, W(cov_j(a))$$

where $cov_j(a)$ is the clauses satisfied by exactly j variables

$$SAT = W(a) \leq \Phi(a) \leq \Phi(a^*) \leq H(k)\, W(a^*) = H(k)\, OPT$$

PoS

C_0	$\overline{x_1} \vee \overline{x_2} \vee \cdots \vee \overline{x_k}$	$w_0 = 1 + \varepsilon$
C_1	x_1	$w_1 = 1$
C_2	x_2	$w_2 = 1/2$
$\vdots$	$\vdots$	$\vdots$
C_j	x_j	$w_j = 1/j$
$\vdots$	$\vdots$	$\vdots$
C_k	x_k	$w_k = 1/k$

Conclusion & future work

PoA and PoS of the MAX SAT and MIN SAT games

Matching lower and upper bounds on the PoA

Connection with non oblivious local search

Other prices: Strong price of anarchy

Explore other paradigmatic problems in CO:

- set cover
- max cut
- spanning tree
- etc

Questions?

Thank you for your attention