Evolutionary game dynamics with migration

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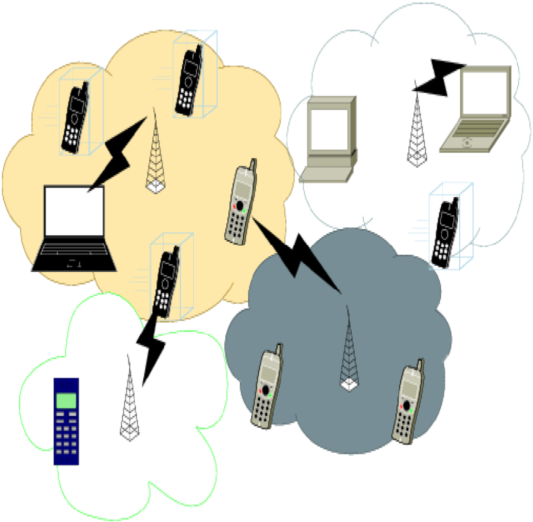
1 -Outline

Association between technologies/interfaces



Evolutionary Games

Evolutionary Games

Consider the following evolutionary game model consisting of :

- Several class of players $\mathcal{E} = \{1, 2, \ldots\}$.
- A *large* number of players (to guarantee non-atomicity),
- Many local and simultaneous interactions between some (possibility random number) of players.
- Each local interaction is described as follows:
  - region or resource called *primary action* ($r \in \mathcal{R}$)
  - each player of each class have a finite set of *secondary actions*

    $$\mathcal{A}_e = \{(r, a) \mid r \in \mathcal{R}, a \in \mathcal{A}_e^r\}, X_e^r = \Delta_{m_e^r}(\mathcal{A}_e^r)$$

    where $\mathcal{A}_e^r$ is the secondary actions in region r for e.
  - a payoff (reward,fitness) function:

    $$F_e : \prod_{e \in \mathcal{E}} \prod_{r \in \mathcal{R}} X_e^r \longrightarrow \mathbb{R}^{|\mathcal{A}_e|}, \quad F_e(x) = (F_{e,a}^r(x))_{r,a}.$$

- the system evolves under some evolutionary game

Objectives

- Solution concepts: Global Equilibrium, Global Neutrally Equilibrium, Choice Constrained Equilibrium, Global Evolutionarily Stable Strategy.
- Multicomponent evolutionary game dynamics, Dynamics with migration constraints.
- Convergence ??? local,global, asymptotic, cycle etc, under some class of evolutionary game dynamics with migration for some class of population games.

Some terms and notations

- *migration*: If a player changes its actions from region r to $\bar{r}$(physically at the same)) or moves from r to $\bar{r}$ (mobility), we say that player migrates from r to $\bar{r}$.
- *migration constraints:* we assume that player from class e in region r can migrate only in the "neighboring regions" $\mathcal{N}_{e,(r,a)} \subseteq \bigcup_r \{\{r\} \times A_e^r\}$ in one-hop.
- the neighborhood set $\mathcal{N}_{e,(r,a)}$ have the property that:

$$(\bar{r}, b) \in \mathcal{N}_{e,(r,a)} \Longleftrightarrow (r, a) \in \mathcal{N}_{e,(\bar{r},b)}$$

Examples of migration/mobility constraints

- *Unidirectional constraint*: the player can change only one of the two components of strategy.

$$\mathcal{N}_{e,(r,a)} = (R \times \{a\}) \bigcup A_e^r$$

- $\mathcal{N}_{e,(r,a)} = R \times A_e$ (free migration).
- $\mathcal{N}_{e,(r,a)} = A_e^r$ (standard, no migration).

Concept of solution: refinement of equilibria

ESS: Evolutionary Stable Strategy (Maynard Smith & Price, 1973).

Definition

(GESS) A strategy x is global evolutionary stable if for every strategy $mut \neq x$, *there exists* $\epsilon_{mut} > 0$ *such that* $\forall e,\ \sum_r \sum_a (x^r_{e,a} - mut^r_{e,a}) F^r_{e,a}(\epsilon\ mut + (1-\epsilon)x) > 0,\ \forall \epsilon \in (0, \epsilon_{mut})$

Robustness, refinement and equilibrium selection: ESS $\Longrightarrow$ proper equilibrium(Myerson), Nash equilibrium, the converse is not always true. See Maynard Smith(1982), Weibull(1995), Hofbauer & Sigmund(1998).

Choice constrained equilibrium

A strategy x is a *choice constrained equilibrium* (CCE) if for all e, r and a such that $x^r_{e,a} > 0$ one has,

$$F^r_{e,a}(x) = \max_{(\bar{r},b)\in\mathcal{N}_{e,(r,a)}} F^{\bar{r}}_{e,b}(x)$$

This corresponds to "local equilibrium" under migration constraints. If $x^r_{e,a} = 0$, we say that (r, a) is not used in class e.

Evolutionary Game Dynamics with migration

Let $\beta_{e,(r,a)}^{(\bar{r},b)}(.)$ be a rule of actions' choice called *revision protocol* (Sandholm,2007). β depends on state of all the population and the payoffs. $\beta_{e,(r,a)}^{(\bar{r},b)}$ conditional switch rate from the secondary action b in region $\bar{r}$ to the secondary action a in region r.

Inflow the action (r, a) of class e

$$\sum_{(\bar{r},b)} x_{e,b}^{\bar{r}} \beta_{e,(r,a)}^{(\bar{r},b)}(x, F(x))$$

Outflow of the action (r, a) of class e:

$$x_{e,a}^{r} \sum_{(\bar{r},b)} \beta_{e,(\bar{r},b)}^{(r,a)}(x, F(x))$$

Let

$$V_{e,F}^{(r,a)}(x) = \sum_{(\bar{r},b)} x_{e,b}^{\bar{r}} \beta_{e,(r,a)}^{(\bar{r},b)}(x, F(x)) - x_{e,a}^{r} \sum_{(\bar{r},b)} \beta_{e,(\bar{r},b)}^{(r,a)}(x, F(x))$$

Assume that the revision protocol satisfies

$$\beta_{e,(\bar{r},b)}^{(r,a)}(x, F(x)) > 0 \Longrightarrow (\bar{r}, b) \in \mathcal{N}_{e,(r,a)}.$$

The game dynamics is then given by

$$\frac{d}{dt} x_{e,a}^{r}(t) = V_{e,F}^{(r,a)}(x(t)).$$

Evolutionary game dynamics with migration: examples

- Replicator dynamics (Taylor & Jonker, 1978) $\beta^{(\bar{r},b)}_{e,(r,a)} = x^r_{e,a}\max(0, F^r_{e,a} - F^{\bar{r}}_{e,b})$
- Brown-von Neumann-Nash (BNN)(1950) dynamics $\beta^{(\bar{r},b)}_{e,(r,a)} = \max(0, F^r_{e,a} - \sum_{(\bar{r},b)} x^r_{e,b}F^{\bar{r}}_{e,b})$
- Smith dynamics $\beta^{(\bar{r},b)}_{e,(r,a)} = \max(0, F^r_{e,a} - F^{\bar{r}}_{e,b})$

Other dynamics: fictitious play, gradient, projection, differential inclusion, best response BR: $BR_e(x) = \arg\max_y\{\sum_{(r,a)} y^r_{e,a}F^r_{e,a}(x)\}$. BR dynamics: $\dot{x}_e \in BR_e(x) - x_e$

Folk theorem (evolutionary version): Every GNE is a stationary of the theses dynamics (in particular for GESS). Every stationary point of BNN, BR or Smith dynamics is GNE of the local game.

Replicator Dynamics with migration

$$\beta^{(\bar{r},b)}_{e,(r,a)} = x^r_{e,a}\max(0, F^r_{e,a} - F^{\bar{r}}_{e,b})$$

$$\frac{d}{dt}x^r_{e,a}(t) = V^{(r,a)}_{e,F}(x(t))$$

$$= \sum_{(\bar{r},b)} x^{\bar{r}}_{e,b}(t)\beta^{(\bar{r},b)}_{e,(r,a)}(x(t), F(x(t))) - x^r_{e,a}(t)\sum^{n}_{(\bar{r},b)} \beta^{(r,a)}_{e,(\bar{r},b)}(x(t), F(x(t)))$$

$$= x^r_{e,a}(t)\left[\sum_{(\bar{r},b)\in\mathcal{N}_{e,(r,a)}} x^{\bar{r}}_{e,b}(t)\left(\max(0, F^r_{e,a} - F^{\bar{r}}_{e,b}) - \max(0, F^{\bar{r}}_{e,b} - F^r_{e,a})\right)\right]$$

$$\frac{d}{dt}x^r_{e,a}(t) = x^r_{e,a}(t)\left[F^r_{e,a} - \sum_{(\bar{r},b)\in\mathcal{N}_{e,(r,a)}} x^{\bar{r}}_{e,b}(t)F^{\bar{r}}_{e,b}\right]$$

Discrete time Replicator Dynamics (DRD)

$$x^r_{e,a}(t+\Delta t) = x^r_{e,a}(t)\frac{1-\Delta_e t+\Delta_e t[\delta_e + F^r_{e,a}(x(t))]}{1-\Delta_e t+\Delta_e t[\delta_e + \sum_{(r,a)} x^r_{e,a}(t)F^r_{e,a}(x(t))]}$$

Note that when $\Delta_e t$ goes to zero, we obtain the continuous time RD.

Discrete Time Replicator Dynamics

$$x^r_{e,a}(t+1) = x^r_{e,a}(t)\frac{\delta_e + F^r_{e,a}(x(t))}{\delta_e + \sum_{(\bar{r},b)\in\mathcal{N}_{e,(r,a)}} x^{\bar{r}}_{e,b}(t)F^{\bar{r}}_{e,b}(x(t))}$$

Every equilibrium of the game is a fixed point of (DRD).

Properties

- Positive Correlation (PC):

$$V_F(x) \neq 0 \Longrightarrow \sum_{e,r,a} [\frac{d}{dt} x^r_{e,a}] F^r_{e,a}(x) > 0.$$

  Under (PC), every CCE is rest point of the dynamics.
- Constrained Nash stationarity (CNS): The rest points of the dynamics are exactly the CCEs of the evolutionary game.

Result

RD, BNN, Smith dynamics satisfy (PC). Moreover, BNN and Smith dynamics satisfy (CNS).

Decomposable dynamics and new hybrid dynamics

the revision protocol β is decomposed in inter-inflow/inter-outflow (resp. intra-inflow/intra-outflow). The migration rates are different from region to a another. The intra-flow is generated by $\rho^r_{e,a}(x)$ and the inter-flow is generated by

$$\begin{cases} \eta^{(r,a)}_{e,(\bar{r},b)}(x) & \text{if } \bar{r} \neq r, (\bar{r}, b) \in \mathcal{N}_{e,(r,a)} \\ 0 & \text{otherwise} \end{cases}$$

Examples:**hybrid evolutionary game dynamics**
β =Replicator+ BNN, β = BNN+Replicator β =Smith+BNN is (PC),(CNS).
↬ speed of convergence of hybrid dynamics?. stability of CCE?.

Monotonicity properties

Result

Assume that the revision protocol β satisfies

$$\beta^{(r,a)}_{e,(\bar{r},b)}(x) = \begin{cases} >0 & \text{if } F^r_{e,a}(x) > F^{\bar{r}}_{e,b}(x),\ (\bar{r},b) \in \mathcal{N}_{e,(r,a)} \\ 0 & \text{otherwise} \end{cases}$$

Then the evolutionary game dynamics with migration generated by β satisfy (PC) and (CNS).

Comment: stationary points of the dynamics coincide with choice constrained equilibria

Elements of proof of (PC)

$$\sum_{e,r,a}[\frac{d}{dt}x^r_{e,a}]F^r_{e,a}(x) = \sum_{e,r,a} V^{(r,a)}_{e,F}(x)F^r_{e,a}(x)$$

$$= \sum_{e,(r,a),(\bar{r},b)} x^{\bar{r}}_{e,b}\beta^{(\bar{r},b)}_{e,(r,a)}F^r_{e,a}(x) - \sum_{e,(r,a),(\bar{r},b)} x^r_{e,a}\beta^{(r,a)}_{e,(\bar{r},b)}F^r_{e,a}(x)$$

$$= \sum_{e,(r,a),(\bar{r},b)} x^{\bar{r}}_{e,b}\underbrace{\beta^{(\bar{r},b)}_{e,(r,a)}[F^r_{e,a}(x) - F^{\bar{r}}_{e,b}(x)]}_{\geq 0}$$

Note that the concavity/convexity properties is not needed here.

Constrained Nash stationary (CNS)

At a CCE, we have that, for all class e,

$$x^r_{e,a}\beta^{(r,a)}_{e,(\bar{r},b)}(x) = 0$$

This implies that

$$V^{(r,a)}_{e,F}(x) = 0, \ \forall e, r, a.$$

Particular class of population games

- "full" potential population games: F has a potential function. Advantages: Every local maximizer of the potential function is a global equilibrium. convergence properties.
  The atomic formulation: (Moderer & Shapley, 1996), population game formulation: (Sandholm(2001)).
- Stable population games (Hofbauer & Sandholm(2006)): F is monotone (in vector-valued sense). Advantages: the set of equilibrium is convex. convergence under some particular dynamics.

Others classes: Supermodular population games. See Topkis(1979) for atomic games. Advantages: monotonicity of the set-valued function (correspondence) BR under the stochastic order on the simplex.

Global convergence in potential games

Result

Global convergence holds in potential games under (PC)

Some elements: let f be a C^1 function such that

$$\nabla f = F$$

then,

$$\frac{d}{dt}f(x(t)) = \sum_e \sum_r \sum_a V_{e,F}^{(r,a)}(x(t)) F_{e,a}^r(x(t))$$

and Lyapunov's theorem completes the proof.

Simple class of dynamics

Result

Every GNE is GNESS in stable games. The set of CCE is asymptotically globally stable in stable games under the dynamics generated by

$$\beta_{e,(r,a)}^{(\bar{r},b)}(x) = \mu_e^r \max\{0, F_{e,a}^r(x) - F_{e,b}^{\bar{r}}(x)\}^\theta,$$

if $(\bar{r}, b) \in \mathcal{N}_{e,(r,a)}$ *and* 0 *otherwise.* μ_e^r *is growth parameter.*

Proof:

$$B(x) = \frac{1}{1+\theta} \sum_{e=1}^{E} \sum_{(r,a)} \sum_{(\bar{r},b)} \mu_e^r x_{e,a}^r \max\{0, F_{e,a}^r(x) - F_{e,b}^{\bar{r}}(x)\}^{1+\theta}$$

- the zeros of B are CCEs.
- since ${}^t\dot{x} D_x F \dot{x} \leq 0$, $\frac{d}{dt} B(x) \leq 0$

Concave function (recall)

$f\,:\,\mathbb{R}^n \longrightarrow \mathbb{R}$ C^1 concave function

- then f satisfies

$$\langle \nabla_y f, x - y \rangle \geq f(x) - f(y)$$

Swap x and y and sum:

$$\langle \nabla_x f, y - x \rangle \geq f(y) - f(x)$$

$$\langle x - y, -\nabla_x f + \nabla_y f \rangle \geq 0$$

$$\langle x - y, \nabla_x f - \nabla_y f \rangle \leq 0$$

link with Rosen's(1965) condition

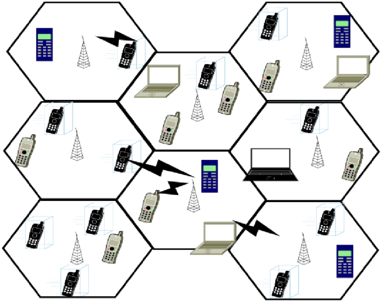
- a potential concave population game is a stable game.

$$\langle x - y, \nabla_x f - \nabla_y f \rangle \leq 0$$

- f is Diagonally Strictly Concave iff f is a strict stable game
  - the game has unique GNE.
  - the game has a unique GESS (equal to GNE).

Application I: *hybrid* power control in wireless communications

- large population of mobile terminals (players),
- distributed base stations in multi-cell CDMA
- each mobile connects to a base station which it chooses from of the set of base stations with an uplink power level from the set of powers.



Interference measurement in **hybrid** power control: CDMA

- payoff: throughput-loss
- throughput, capacity is expressed as function of the inter-interference and the intra-interference.
- loss: function of the power consumption
- **migration**: $\mathcal{N}_{e,(r,a)}$ is the reachable regions (base stations, resources) with the power a. Example:

$$\{r, \text{throughput}^r(.) \geq \nu_{\min}\}$$

$\nu_{\min}$ = minimum requirement QoS-constraint.

Interference measurement in **hybrid** power control in OFDMA-based Networks

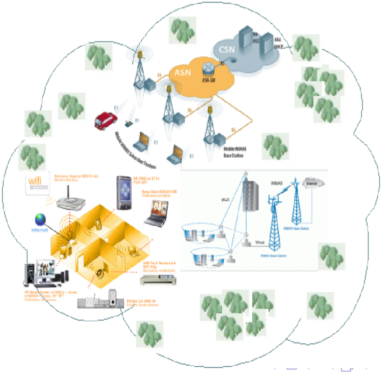
- Several cells
- Orthogonal Frequency-Division Multiple Access: interference from the other cells (inter-cells) in the neighborhood
$$F^r_{e,a}(x) = F^r_{e,a}(x^{-r})$$
- it can be seen as a stable game : the performance decreases when the interference from the others cells in $\mathcal{N}$ increasing. Hence, convergence under the **pairwise comparison evolutionary game dynamics**.

Example:$\theta > 0$

$$\beta^{(\bar{r},b)}_{e,(r,a)}(x) = \mu^r_e \begin{cases} \max\{0, F^r_{e,a}(x) - F^{\bar{r}}_{e,b}(x)\}^{\theta} & \text{if } (\bar{r}, b) \in \mathcal{N}_{e,(r,a)} \\ 0 & \text{otherwise.} \end{cases}$$

Application II: Multihoming and association problems in heterogenous networks

- several technologies and interfaces such as UMTS, WLAN, WiMax etc.
- class: group of mobiles with the same technologies ...
- geographical regions



Performance of the system: operator

- The migration (hybrid) model increases the global payoff of the system

$$x \longmapsto \sum_{e,r,a} x^r_{e,a} F^r_{e,a}(x)$$

- $\looparrowright$ the total throughput is improved by adding "migration".

Two-stage local game: Stackelberg approach

- each technology or class is managed by some operator.
- *first round:* the operator chooses some pricing function.
- *second round*: the second level players (followers) react (they know the price of services).
- The **pricing function** evolves in time. (in the standard model the price is fixed!).
- solution by **backward induction**, iterative algorithms for the operator to update the prices and evolutionary game dynamics with multicomponent strategies for the followers to converge to choice constrained equilibrium.

Conclusions

- we have developed evolutionary game dynamics with multicomponent strategies of large population with several class or types of players.
- we have studied convergence to *choice constrained equilibria* under positive correlation (PC) and constrained Nash stationarity (CNS) for some class of dynamics.
- global convergence for potential population games and stable population games.
- Application I: *hybrid* power control in wireless communications.
  - CDMA
  - OFDMA
- Application II: Multihoming and association problems in heterogenous networks.

Some references

- G. Brown and J. Von Neumann. *Solutions of games by differential equations*. In H. Kuhn and A. Tucker, editors, Contributions to the Theory of Games I, Annals of Mathematics Studies 24, pages 73-79. Princeton University Press, 1950.
- J. Hofbauer and K. Sigmund.*Evolutionary game dynamics*. American Mathematical Society, Vol 40 No. 4, pp. 479-519, 2003.
- J. Miękis and T. Patkowski,*Population dynamics with a stable efficient equilibrium,* Journal of Theoretical Biology, Volume 237, Issue 4, 21 December 2005, Pages 363-368.
- J. Maynard Smith and GR. Price. *The logic of animal conflict*, Nature 246, 15-18. 1973.
- William H. Sandholm, *Population Games and Evolutionary Dynamics*, 2008, MIT Press.
- P. Taylor and L. Jonker. *Evolutionary stable strategies and game dynamics*, Mathematical Biosciences, 16:76-83, 1978.

The end!

Delayed hybrid evolutionary game dynamics

Suppose now that payoffs are not instantaneously: an action $(r, a$ taken today (date t) by a player from e, will have its effect only $t+\tau^r_{e,a}$ times later. The payoffs are delayed. Denote by

$$\bar{F}^r_{e,a,\tau}(x(t)) = \sum_{(\bar{r},b)\in\mathcal{N}_{e,(r,a)}} x^{\bar{r}}_{e,b}(t) F^r_{e,a}(x(t-\tau^r_{e,a}))$$

Delayed Replicator Dynamics with migration

$$\frac{d}{dt}x^r_{e,a} = \mu^r_e x^r_{e,a}(t)\left[F^r_{e,a} - \bar{F}^r_{e,a,\tau}(x(t))\right]$$

Delayed hybrid evolutionary game dynamics(continued)

Delayed Replicator Dynamics with migration

$$\frac{d}{dt}x^r_{e,a} = \mu^r_e x^r_{e,a}(t)\left[F^r_{e,a}(x(t)) - \bar{F}^r_{e,a,\tau}(x(t))\right]$$

Result

(i) A choice constrained equilibrium can be unstable for large delays (this occurs also in stable games).
(ii) Let x^* be an GNE. If

$$\left(\max_{e,r,a} \tau^r_{e,a}\right) \parallel \nabla_{x^*} F \parallel_\infty < 1$$

then the x^* is asymptotically stable.

How to stabilize the system?

Result

(iii) Given τ, we are able to find

$$\mu' = \mu_e^r(\{\tau_{e',b}^{\bar{r}}\}_{e',\bar{r},b})$$

such that x^* becomes asymptotically stable under the delayed replicator dynamics with migration with the updated growth μ'

.

(iv) this operation does not work in non-regular game dynamics such as imitate the better dynamics, Smith dynamics where the CCE are unstable for any positive delay.