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Learning and Traffic Equilibrium

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OUTLINE

A) Motivation & Questions

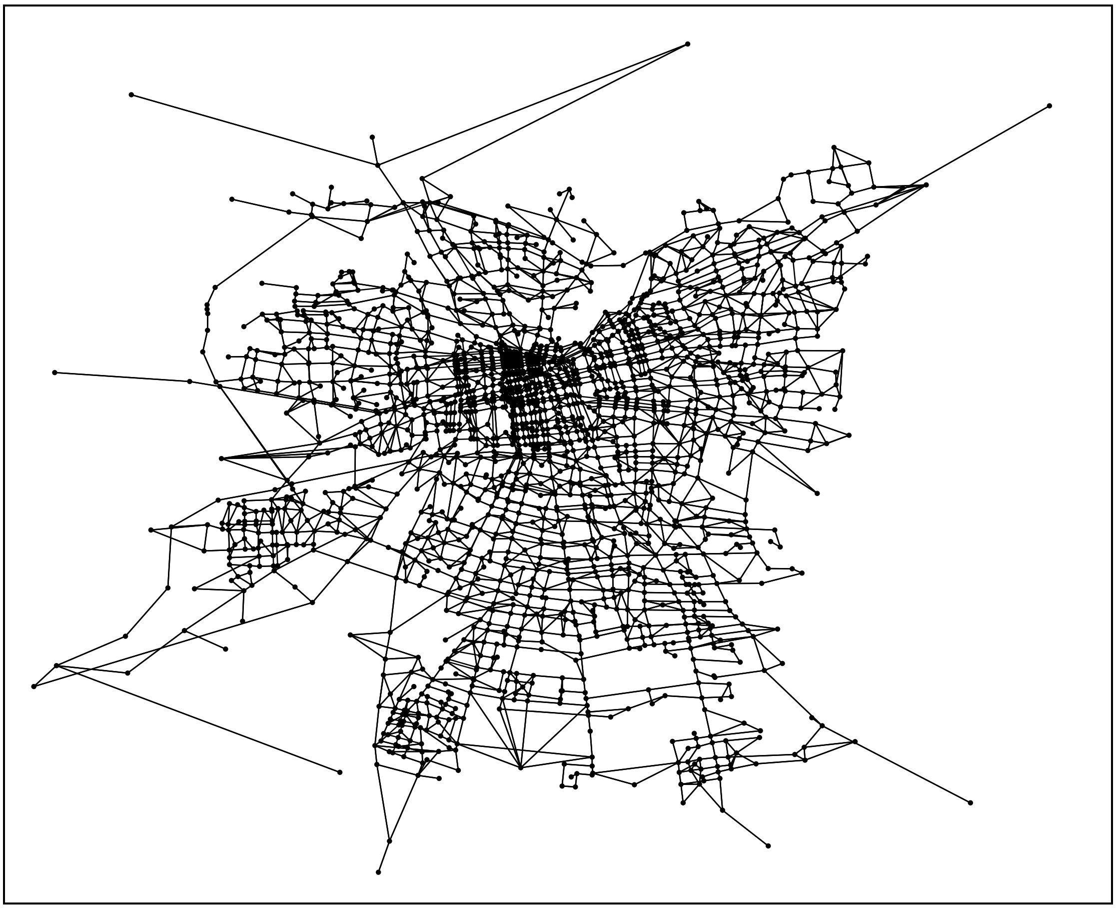
1) Method of successive averages

2) Markovian traffic equilibrium

3) Learning and network equilibrium

A) Motivation & Questions

SANTIAGO NETWORK



2266 nodes / 7636 arcs / 409 destinations

$\Rightarrow$ over 3 million flow variables

SANTIAGO: some figures

6	million people
1	million cars
50	thousand taxis
6	thousand buses
4	metro lines

SANTIAGO: some figures

Mode	Daily trips	%
Bus	4.350.000	53.7
Car	1.750.000	21.7
Taxi	820.000	10.1
Metro	800.000	9.9
Other	370.000	4.6
Total	8.090.000	100.0

+ 3 million non-motorized trips

SANTIAGO: some figures

Morning peak
∼20% of trips
∼500.000 car+taxi trips
∼29.000 OD pairs

Network user equilibrium (Wardrop'52)

Data
$$\left\{ \begin{array}{ll} G = (N, A) & \text{network} \\ t_a = s_a(w_a) & \text{arc travel time} \\ g_i^d & \text{demand} \\ \mathcal{R}_i^d & \text{routes} \end{array} \right.$$

Network user equilibrium (Wardrop'52)

Data $\begin{cases} G = (N, A) & \text{network} \\ t_a = s_a(w_a) & \text{arc travel time} \\ g_i^d & \text{demand} \\ \mathcal{R}_i^d & \text{routes} \end{cases}$

Decompose $g_i^d = \sum_{r \in \mathcal{R}_i^d} x_r$ so that

$$[x_r \geq 0] \quad \& \quad [x_r > 0 \Rightarrow T_r = \tau_i^d]$$

where

$T_r = \sum_{a \in r} s_a(w_a)$ (route times)

$w_a = \sum_{r \ni a} x_r$ (total arc flows)

$\tau_i^d = \min_{r \in \mathcal{R}_i^d} T_r$ (minimal time)

Variational formulation

(Beckman-McGuire-Winsten'56)

$$(P) \quad \begin{cases} \underset{w,v}{\text{Min}} \quad \sum_a \int_0^{w_a} s_a(x)\,dx \\ \text{s.t.} \\ w_a = \sum_d v_a^d \\ g_i^d + \sum_{A_i^-} v_a^d = \sum_{A_i^+} v_a^d \\ v_a^d \geq 0 \end{cases}$$

$$\Downarrow$$

There exists a unique equilibrium w^*

Dual formulation (Fukushima'84)

Change of variables: $w_a \leftrightarrow t_a$

$$(D) \qquad \operatorname*{Min}_{t} \quad \underbrace{\sum_a \int_0^{t_a} s_a^{-1}(x)\,dx - \sum_{i,d} g_i^d \tau_i^d(t)}_{\substack{\phi(t) \\ \text{strictly convex}}}$$

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$t \mapsto \tau_i^d(t)$ concave non-smooth polyhedral solution of Bellman's equations

$$\begin{cases} \tau_d^d = 0 \\ \tau_i^d = \min_{a \in A_i^+} [t_a + \tau_{j_a}^d] \end{cases}$$

Method of Successive Averages

- Compute $t_a^k = t_a(w_a^k)$

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(MSA) $$\boxed{w^{k+1} = (1 - \lambda_k) w^k + \lambda_k \Phi(w^k)}$$

Equilibrium $\Leftrightarrow$ fixed point of $w \stackrel{\Phi}{\mapsto} \tilde{w}$

Case Φ non-expansive: ($>$200 refs in MathSciNet)

Mann'53, Krasnoselskii'55, Edelstein'66,...

The problem is... our $\Phi(\cdot)$ is expansive!!!

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The problem is... our $\Phi(\cdot)$ is expansive!!!

Questions:

1. Why then MSA works?
2. What if users are heterogeneous?
3. How is equilibrium attained?

1) Method of successive averages

MSA revisited

$$(D) \qquad \underset{t}{\text{Min}} \quad \underbrace{\sum_a \int_0^{t_a} s_a^{-1}(x)\,dx - \sum_{i,d} g_i^d \tau_i^d(t)}_{\phi(t)}$$

MSA revisited

$$(D) \qquad \underset{t}{\text{Min}} \quad \underbrace{\sum_a \int_0^{t_a} s_a^{-1}(x)\,dx - \sum_{i,d} g_i^d \tau_i^d(t)}_{\phi(t)}$$

Back to w... the equilibrium w^* solves

$$(\tilde{D}) \qquad \underset{w}{\text{Min}} \;\; \psi(w) := \phi(s_a(w_a) : a \in A)$$

and we may re-write

(MSA) $$\boxed{\frac{w^{k+1} - w^k}{\lambda_k} \in -D(w^k)^{-1}\partial\psi(w^k)}$$

$D(w) = \text{diag}(s_a'(w_a))$ is part of "$\nabla^2\psi(w)$"

MSA works

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MSA works

(MSA) $$\frac{w^{k+1} - w^k}{\lambda_k} \in -D(w^k)^{-1} \partial\psi(w^k)$$

(RGF) $$\frac{dw}{dt} \in -D(w)^{-1} \partial\psi(w)$$

(Alvarez-Bolte-Brahic'04)

Theorem. If $\sum \lambda_k = \infty$ and $\sum \lambda_k^2 < \infty$ then

$$w^k \to w^* \in \text{Argmin } \psi$$

2) Markovian traffic equilibrium

Heterogeneous users

$$\left.\begin{aligned}\tilde{t}_a &= t_a + \epsilon_a \\ \tilde{T}_r &= \textstyle\sum_{a \in r} \tilde{t}_a \\ \tilde{\tau}_i^d &= \min_{r \in \mathcal{R}_i^d} \tilde{T}_r\end{aligned}\right\} \text{random variables}$$

Heterogeneous users

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At every intermediate node i in his trip a user selects a *random* optimal arc

$$\underset{a \in A_i^+}{\text{Argmin}}\ \tilde{t}_a + \tilde{\tau}_{j_a}^d$$

$$\Downarrow$$

Markov chain for each destination d

Markovian equilibrium

Expected in-flow

$$x_i^d = g_i^d + \sum_{a \in A_i^-} v_a^d$$

leaves node i according to

$$v_a^d = x_i^d \, \mathbb{P}(\tilde{t}_a + \tilde{\tau}_{j_a}^d \leq \tilde{t}_b + \tilde{\tau}_{j_b}^d \; \forall \, b \in A_i^+)$$

with

$$t_a = s_a(w_a)$$

$$w_a = \sum_d v_a^d$$

Discrete choice models

$$\varphi(u) = \mathbb{E}[\min\{u_1 + \varepsilon_1, \ldots, u_n + \varepsilon_n\}]$$

$$\pi_i = \mathbb{P}(u_i + \varepsilon_i \text{ is optimal}) = \frac{\partial \varphi}{\partial u_i}(u)$$

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Example: Multinomial LOGIT

(errors $\sim$ iid Gumbel – extremal distribution)

$$\begin{aligned} \varphi(u) &= -\tfrac{1}{\beta} \ln[\textstyle\sum_j \exp(-\beta u_j)] \\ \pi_i &= \frac{\exp(-\beta u_i)}{\sum_j \exp(-\beta u_j)} \end{aligned}$$

Variational formulation

$\tau_i^d = \mathbb{E}(\tilde{\tau}_i^d)$ is the unique solution of the stochastic Bellman's equations

$$\begin{cases} \tau_d^d = 0 \\ \tau_i^d = \mathbb{E}(\min_{a \in A_i^+} [t_a + \tau_{j_a}^d + \epsilon_a^d]) \end{cases}$$

$t \mapsto \tau_i^d(t)$ concave & smooth.

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$t \mapsto \tau_i^d(t)$ concave & smooth.

Stochastic equilibrium characterized as

$$(D) \quad \min_t \ \phi(t) = \sum_a \int_0^{t_a} s_a^{-1}(x)\,dx - \sum_{i,d} g_i^d \tau_i^d(t)$$

...identical to deterministic case!

Stochastic MSA

- Compute $t_a^k = t_a(w_a^k)$
- Solve stochastic Bellman's equations
- Compute Markov chain flows v_a^d
- Aggregate $\tilde{w}_a^k = \sum_d v_a^d$
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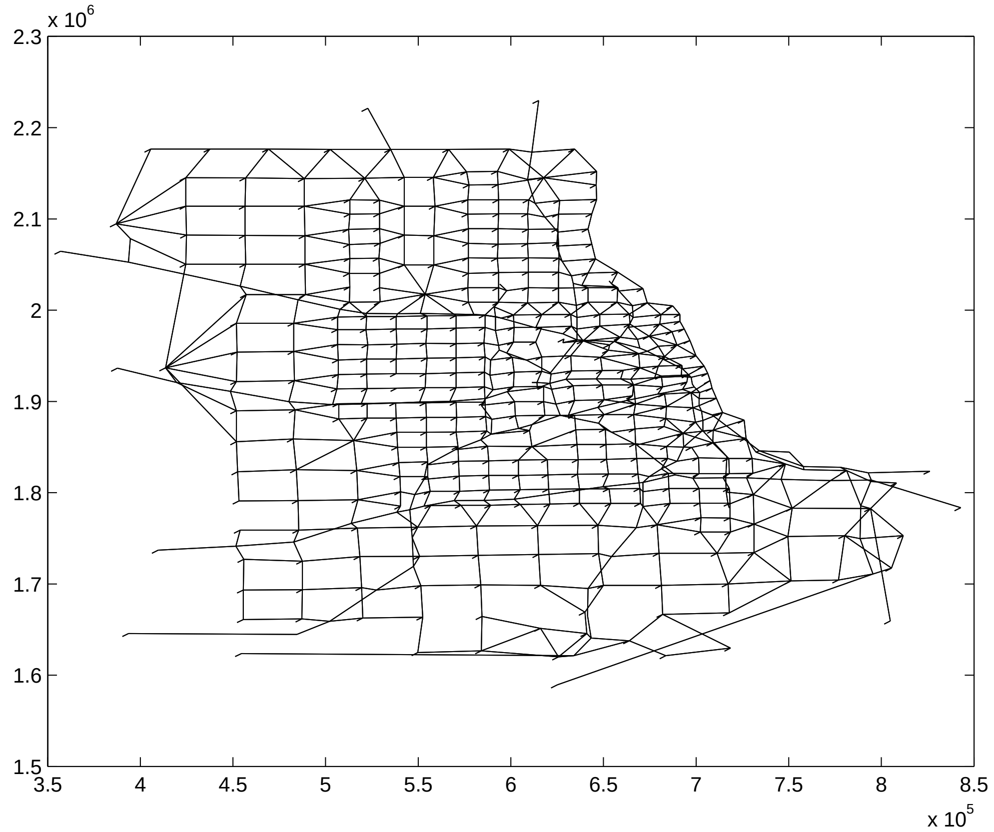
May be rewritten as

(SMSA) $$\frac{w^{k+1} - w^k}{\lambda_k} = -D(w^k)^{-1} \nabla \psi(w^k)$$

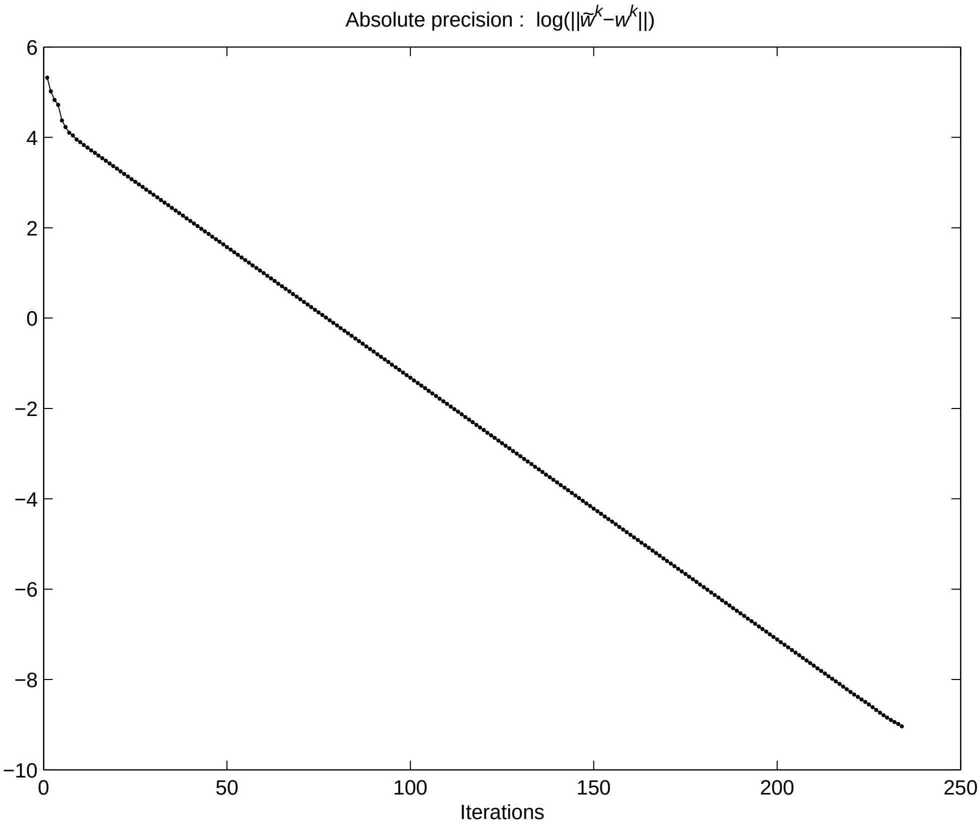
and converges to stochastic equilibrium w^*

Chicago Network



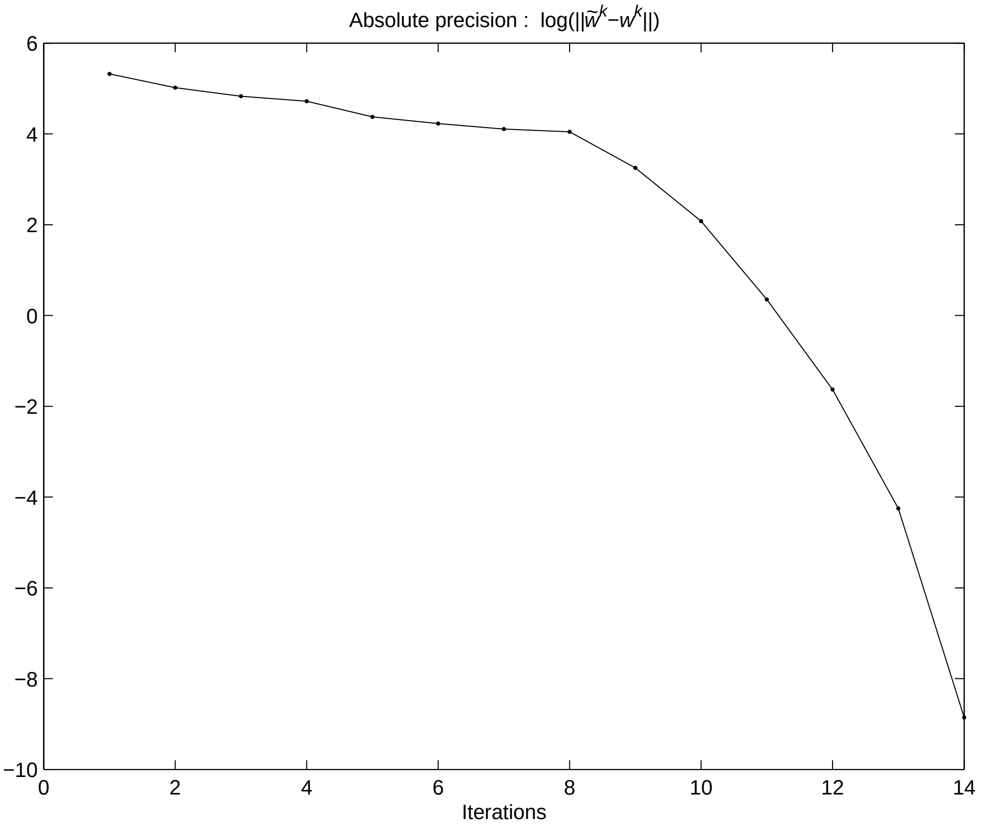
933 nodes / 2950 arcs

SMSA iterations



Execution time (1.6Mhz Pentium): 29[min]

SMSA-Newton iteracions



Execution time (1.6Mhz Pentium): 11[min]

3) Learning and equilibrium

Data & Variables

$i \quad = \quad 1, \ldots, N$ drivers

$r \quad = \quad 1, \ldots, M$ routes

$c_u^r \quad = \quad$ cost of route r with u drivers

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$i \quad = \quad 1, \ldots, N$ drivers

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$c_u^r \quad = \quad$ cost of route r with u drivers

$x^{ir} \quad = \quad$ perception of driver i on route r

$\pi^{ir} \quad = \quad$ pbb driver i to choose route r

$$\pi^{ir} = \frac{\exp(-\beta x^{ir})}{\sum_\ell \exp(-\beta x^{i\ell})}$$

Learning process

Learning process

x^{ir}_{n-1}
state

Learning process

$x_{n-1}^{ir} \rightsquigarrow \pi_n^{ir}$

state pbb's

Learning process

$$x_{n-1}^{ir} \rightsquigarrow \pi_n^{ir} \rightsquigarrow r_n^i$$

state pbb's routes

Learning process

x_{n-1}^{ir}	$\rightsquigarrow$	π_n^{ir}	$\rightsquigarrow$	r_n^i	$\rightsquigarrow$	u_n^r
state		pbb's		routes		loads

Learning process

$$x_{n-1}^{ir} \rightsquigarrow \pi_n^{ir} \rightsquigarrow r_n^i \rightsquigarrow u_n^r \rightsquigarrow c_n^r$$

state	pbb's	routes	loads	costs

Learning process

x_{n-1}^{ir}	$\rightsquigarrow$	π_n^{ir}	$\rightsquigarrow$	r_n^i	$\rightsquigarrow$	u_n^r	$\rightsquigarrow$	c_n^r	$\rightsquigarrow$	x_n^{ir}
state		pbb's		routes		loads		costs		update

Learning process

x_{n-1}^{ir}	$\rightsquigarrow$	π_n^{ir}	$\rightsquigarrow$	r_n^i	$\rightsquigarrow$	u_n^r	$\rightsquigarrow$	c_n^r	$\rightsquigarrow$	x_n^{ir}
state		pbb's		routes		loads		costs		update

$$x_n^{ir} = \begin{cases} (1-\lambda_n)x_{n-1}^{ir} + \lambda_n c_n^r & \text{if } r = r_n^i \\ x_{n-1}^{ir} & \text{if } r \neq r_n^i \end{cases}$$

Learning process

$$x_{n-1}^{ir} \rightsquigarrow \pi_n^{ir} \rightsquigarrow r_n^i \rightsquigarrow u_n^r \rightsquigarrow c_n^r \rightsquigarrow x_n^{ir}$$

state pbb's routes loads costs update

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$$\boxed{x_n - x_{n-1} = \lambda_n[\tilde{c}_n - x_{n-1}]}$$

Convergence to an equilibrium?

Adaptive dynamics

$$\frac{x_n - x_{n-1}}{\lambda_n} = \tilde{c}_n - x_{n-1}$$

a.s. convergence to an ICT of

$$\frac{dx}{dt} = \mathbb{E}(\tilde{c}|x) - x$$

Ljung'77, Benaim-Hirsch'96, Benaim-Hofbauer-Sorin'05

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$$\boxed{\frac{dx^{ir}}{dt} = \pi^{ir}(x)[C^{ir}(x) - x^{ir}]}$$

$$\begin{aligned} C^{ir}(x) &= \mathbb{E}(c^r_\bullet | i \text{ chooses } r) \\ \pi^{ir}(x) &= \frac{\exp(-\beta x^{ir})}{\sum_\ell \exp(-\beta x^{i\ell})} \end{aligned}$$

Adaptive dynamics... cont'd

$C^{ir}(x) = \mathbb{E}(c^r_\bullet | i \text{ chooses } r)$

Adaptive dynamics... cont'd

$C^{ir}(x) = \mathbb{E}(c^r_\bullet | i \text{ chooses } r)$

$$\begin{aligned} C^{ir}(x) &= F^{ir}(\Pi(x)) \\ \Pi(x) &= (\pi^{ir}(x)) \end{aligned}$$

Adaptive dynamics... cont'd

$C^{ir}(x) = \mathbb{E}(c^r_\bullet | i \text{ chooses } r)$

$$C^{ir}(x) = F^{ir}(\Pi(x))$$
$$\Pi(x) = (\pi^{ir}(x))$$

$F^{ir}(\pi) = \mathbb{E}[c^r_{U^r} | X^{ir} = 1]$

$U^r = \sum_j X^{jr}$

X^{ir} Bernoulli r.v. $\mathbb{P}(X^{ir}{=}1) = \pi^{ir}$

Example: 2 drivers $\times$ 2 routes

$$\frac{dx^{1a}}{dt} = \pi^a(x^1)[C^a(x^2) - x^{1a}] \qquad \text{(driver 1)}$$
$$\frac{dx^{1b}}{dt} = \pi^b(x^1)[C^b(x^2) - x^{1b}]$$

$$\frac{dx^{2a}}{dt} = \pi^a(x^2)[C^a(x^1) - x^{2a}] \qquad \text{(driver 2)}$$
$$\frac{dx^{2b}}{dt} = \pi^b(x^2)[C^b(x^1) - x^{2b}]$$

$$\pi^a(x) = \exp(-\beta x^a)/[\exp(-\beta x^a) + \exp(-\beta x^b)]$$
$$\pi^b(x) = \exp(-\beta x^b)/[\exp(-\beta x^a) + \exp(-\beta x^b)]$$

$$C^a(x) = c_1^a \pi^b(x) + c_2^a \pi^a(x)$$
$$C^b(x) = c_1^b \pi^a(x) + c_2^b \pi^b(x)$$

Adaptive dynamics: simulation

$$\frac{dx^{ir}}{dt} = \pi^{ir}(x)[C^{ir}(x) - x^{ir}]$$

Rest points: definition

$$\frac{dx^{ir}}{dt} = \pi^{ir}(x)[C^{ir}(x) - x^{ir}]$$

$$\mathcal{E} = \{\text{rest points}\} = \{x : x^{ir} = C^{ir}(x)\}$$

Rest points: definition

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$$x = F(\Pi(x))$$

$$\begin{cases} x = F(\pi) \\ \pi = \Pi(x) \end{cases}$$

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$$x = F(\Pi(x))$$

$$\begin{cases} x = F(\pi) \\ \pi = \Pi(x) \end{cases}$$

$x \rightleftarrows \pi$ bijection on $\mathcal{E}$

$$\mathcal{P} = \Pi(\mathcal{E}) = \{\text{rest probabilities}\}$$

Rest points: characterization

N-person game with congestion + entropy

Strategies $\longrightarrow$ $\pi^i \in \Delta(R)$

Payoffs $\longrightarrow$ $G^i(\pi) = -\langle \pi^i, F^i(\pi) \rangle - \frac{1}{\beta} \sum_r \pi^{ir} [\ln \pi^{ir} - 1]$

Rest points: characterization

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Payoffs $\rightarrow$ $G^i(\pi) = -\langle \pi^i, F^i(\pi)\rangle - \frac{1}{\beta}\sum_r \pi^{ir}[\ln \pi^{ir} - 1]$

Theorem: $\mathcal{P} = \Pi(\mathcal{E}) = \{\text{Nash equilibria}\}$

Rest points: existence/uniqueness

Theorem:

(a) there exist rest points

(a) exactly one of them is symmetric: $\widehat{x}^{ir} = \widehat{x}^{jr}$

(b) if $\beta\delta < 2$ then $\widehat{x}$ is the only rest point

$$\delta \;=\; \max_{r,u}[c^r_u - c^r_{u-1}] \quad \text{(congestion jump)}$$

Rest points: attractors

Theorem: If $\beta\delta < 2$ then $\widehat{x}$ is a local attractor

Rest points: attractors

Theorem: If $\beta\delta < 2$ then $\widehat{x}$ is a local attractor

Theorem: If $\beta\delta < \frac{2}{N-1}$ then

(a) $\widehat{x}$ is a global attractor

(b) the learning process converges $x_n \stackrel{a.s.}{\longrightarrow} \widehat{x}$.

Potential

$$
\begin{aligned}
F(\pi) &= \nabla H(\pi) \\
H(\pi) &= \sum_r \mathbb{E}(c_1^r + c_2^r + \cdots + c_{U^r}^r)
\end{aligned}
$$

Potential

$$F(\pi) = \nabla H(\pi)$$

$$H(\pi) = \sum_r \mathbb{E}(c_1^r + c_2^r + \cdots + c_{U^r}^r)$$

Lagrangian

$$H_\beta(\pi) = H(\pi) + \frac{1}{\beta}\sum_{ir} \pi^{ir}[\ln(\pi^{ir}) - 1]$$

$$\mathcal{L}(\pi;\lambda) = H_\beta(\pi) - \sum_i \lambda^i[\sum_r \pi^{ir} - 1]$$

Lagrangian dynamics

$$
\begin{aligned}
L(x;\lambda) &= \mathcal{L}(\pi(x,\lambda);\lambda) \\
\pi^{ir}(x,\lambda) &= \mathsf{exp}(-\beta(x^{ir}-\lambda^{i})) \\
\lambda^{i}(x) &= -\frac{1}{\beta}\,\mathsf{ln}(\sum_{r}\mathsf{exp}(-\beta x^{ir}))
\end{aligned}
$$

Lagrangian dynamics

$$L(x;\lambda) = \mathcal{L}(\pi(x,\lambda);\lambda)$$

$$\pi^{ir}(x,\lambda) = \exp(-\beta(x^{ir} - \lambda^i))$$

$$\lambda^i(x) = -\frac{1}{\beta}\ln(\sum_r \exp(-\beta x^{ir}))$$

$$\boxed{\dot{x} = -\frac{1}{\beta}\nabla_x L(x;\lambda(x))}$$

Rest points revisited

For $\pi = \Pi(x)$ the following are equivalent

(a) $x \in \mathcal{E}$

(b) $\nabla_x L(x, \lambda(x)) = 0$

(c) π is a Nash equilibrium

(d) $\nabla_\pi \mathcal{L}(\pi, \lambda) = 0$ for some $\lambda \in \mathbb{R}^M$

(e) π is a critical point of $H_\beta(\cdot)$ on $\Delta(R)^N$

Moreover, if $\beta\delta < 1$ then H_β is strongly convex and $\widehat{\pi} = \Pi(\widehat{x})$ is its unique minimum.