Routing in massively dense ad-hoc networks: Continum equilibrium

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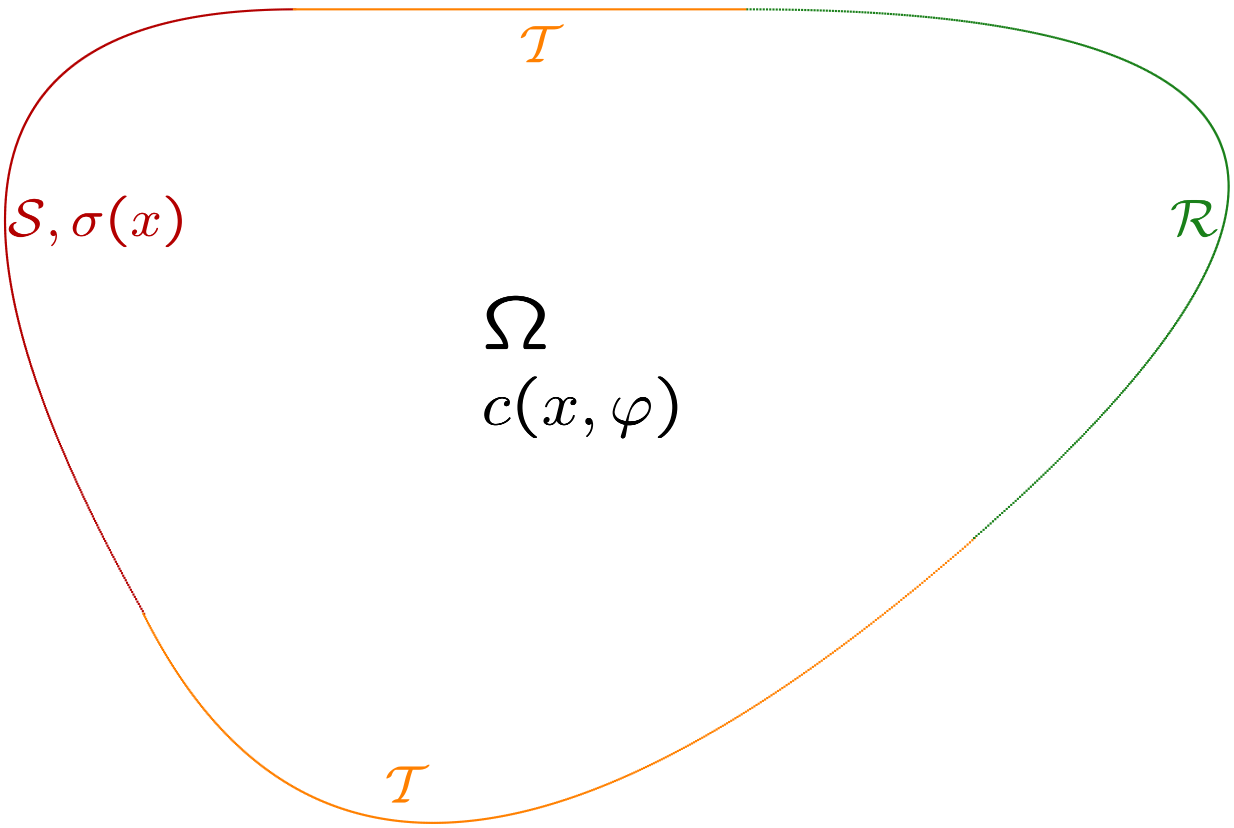
The problem

Let Ω be a connected smooth open set in $\mathbb{R}^2$, with $\partial\Omega = \mathcal{R} \cup \mathcal{S} \cup \mathcal{T}$.

Data


$\mathcal{T}$
$\mathcal{S}, \sigma(x)$
$\mathcal{R}$
Ω
$c(x, \varphi)$
$\mathcal{T}$

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Investigate both the collectively optimal routing and the Wardrop equilibria.

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Will automatically imply

$$\int_{\mathcal{S}} \sigma(x)\,\mathrm{d}s + \int_{\mathcal{R}} \langle n(x), f(x) \rangle\,\mathrm{d}s = 0\,.$$

Mathematical model : costs

Let $e_\theta = (\cos\theta, \sin\theta)$ be the direction of travel of a message.
Total cost incurred in a path from $x(t_0) = x_0 \in \mathcal{R}$ to $x(t_1) = x_1 \in \mathcal{S}$ is

$$J(e_\theta(\cdot)) = \int_{x_0}^{x_1} c\Big(x, \|f(x)\|\Big) \mathrm{d}s = \int_{t_0}^{t_1} c\Big(x(t), \|f(x(t))\|\Big)\, \mathrm{d}t\,.$$

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Let $$C(x,\varphi) := c(x,\varphi)\varphi$$

Total (collective) cost of congestion is

$$G(f(\cdot)) = \int_\Omega c\Big(x, \|f(x)\|\Big)\|f(x)\| \,\mathrm{d}x = \int_\Omega C\Big(x, \varphi(x)\Big) \,\mathrm{d}x\,.$$

Global optimum : necessary conditions

Dualize the constraint $\mathrm{div} f = 0$ with a dual variable $p(x)$, use Green's formula to rewrite lagrangian $\mathcal{L}$. Fréchet derivative $\mathrm{D}_f \mathcal{L}(f^\star, p) = 0$ yields

$$\forall x : f^\star(x) \neq 0\,, \quad \mathrm{D}_2 C(x, \|f^\star(x)\|) \frac{1}{\|f^\star(x)\|} f^\star(x) = \nabla p(x)\,.$$

If $\mathrm{D}_2 C(x, \varphi)/\varphi$ un bounded as $\varphi \to 0$, then at $\varphi = 0$, $0 \in \partial_\varphi \mathcal{L}$ yields

$$\forall x : f^\star(x) = 0\,, \quad \mathrm{D}_2 C(x, 0) \geq \|\nabla p(x)\|\,.$$

Equivalent formulation

Solving the necessary conditions can be stated as : find two scalar functions $p(\cdot)$ and $\varphi(\cdot)$ (in carefully chosen function spaces), such that

$$\begin{array}{ll} \forall x \in \Omega\,, & \|\nabla p(x)\| \leq \mathrm{D}_2 C(x, \varphi(x))\,, \\ \forall x : \varphi(x) \neq 0\,, & \|\nabla p(x)\| = D_2 C(x, \varphi(x))\,, \\ \forall x \in \mathcal{R}\,, & p(x) = 0\,, \end{array}$$

and with
$$f^\star(x) := \frac{\varphi(x)}{\mathrm{D}_2 C(x, \varphi(x))} \nabla p(x)\,,$$

$$\begin{array}{ll} \forall x \in \Omega\,, & \mathrm{div} f^\star(x) = 0\,, \\ \forall x \in \mathcal{Q}\,, & \langle n(x), f^\star(x) \rangle = \sigma(x)\,. \end{array}$$

Wardrop equilibrium

The Hamilton-Jacobi-Carathéodory-Bellman equation of the problem is

$$\begin{aligned} &\forall x \in \Omega\,, \quad \min_\theta \langle e_\theta, \nabla V(x)\rangle + c(x, \|f^\star(x)\|) = 0\,, \\ &\forall x \in \mathcal{R}\,, \quad V(x) = 0\,. \end{aligned}$$

and the optial direction of motion is opposite to ∇V.

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Wherever $\|f^\star(x)\| \neq 0$, this coincides with previous equations upon replacing p by $-\nabla V$ and $\mathrm{D}_2 C$ by c. Hence one can look for the Wardrop equilibrium by solving the global optimization problem with

$$C(x,\varphi) = \int_0^\varphi c(x,\psi)\,\mathrm{d}\psi\,.$$

Generalizes Beckman, 1952, Beckman et al, 1956.

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Theorem If $c(x,\varphi) = c(x)\varphi^\alpha$, $\alpha > 0$, any global equilibrium where $\|f^\star(x)\| \neq 0$ over Ω is a Wardrop equilibrium.

Linear congestion costs

If $c(x,\varphi) = c(x)\varphi$, the equations for p uncouples from φ and becomes a standard mixed Neuman-Dirichlet elliptic PDE:

$$\begin{aligned} &\forall x \in \Omega\,, \quad \mathrm{div}\left(\frac{1}{c(x)}\nabla p(x)\right) = 0\,,\\ &\forall x \in \mathcal{Q}\,, \quad \frac{\partial p}{\partial n}(x) = c(x)\sigma(x)\,,\\ &\forall x \in \mathcal{R}\,, \quad p(x) = 0\,. \end{aligned}$$

⇒ (Existence and) uniqueness. Solution e.g. via standard finite element method.

$$f^\star(x) = \frac{1}{c(x)}\nabla p(x)\,.$$

No congestion, minimize delays

We investigate the case where the cost incurred are only linked to the location, but the network is uncongested. Then $c(x, \varphi) = c(x)$. Now $\mathrm{D}_2 C(x, \varphi) = c(x)$ and $\mathrm{D}_2 C / \varphi \to \infty$ as $\varphi \to 0$.

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Equivalent formulation: Find two scalar functions $p(\cdot)$ and $\psi(\cdot)$ such that

$$\forall x \in \Omega\,,\ \psi(x) \geq 0\,,\ \|\nabla p(x)\| \leq c(x)\,,\ \psi(x)[\|\nabla p(x)\| - c(x)] = 0\,,$$

$$\begin{aligned} \forall x \in \Omega\,, &\quad \psi(x)\Delta p(x) + \langle \nabla\psi(x), \nabla p(x)\rangle = 0\,,\\ \forall x \in \mathcal{Q}\,, &\quad \psi(x)\langle n(x), \nabla p(x)\rangle = \sigma(x)\,. \end{aligned}$$

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We propose an algorithm proved convergent if $f^\star(x) \neq 0\ \forall x \in \Omega$. Proves uniqueness in that case. No satisfactory result for the general case.

Bibliography

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