# Graphs and Path Equilibria

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# Outline

Routing: definitions

Dalograph: definitions

Dalograph: existence of solutions

Routing: existence of solutions

# Routing problem

Let $\Sigma = \{a, b, \dots\}$ be an alphabet. A routing problem consists of:

A binary relation over finite words on the alphabet $\Sigma$, *e.g.* $bb \prec a$.

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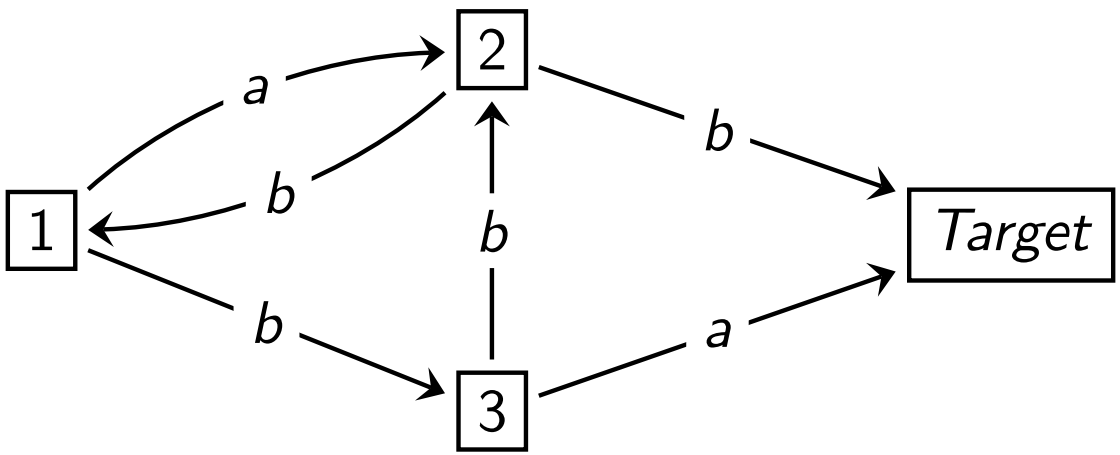
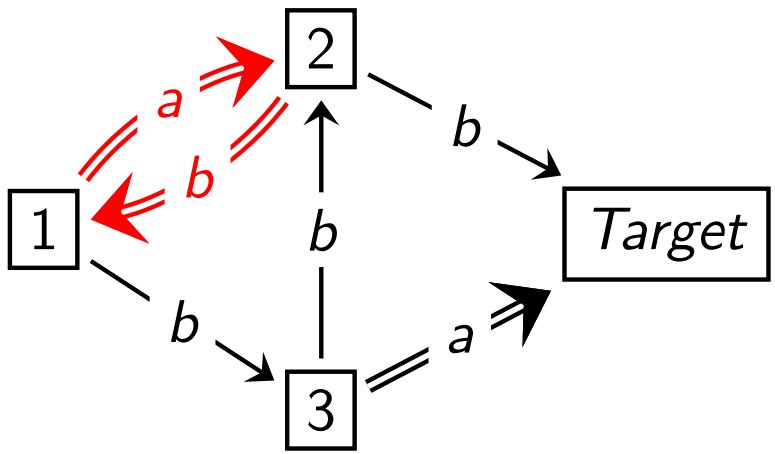
A finite digraph such that:

- Each arc bears a label from $\Sigma$.
- One target node has no outgoing arc.
- Target is reachable from everywhere.
  (Each node but Target has an outgoing arc.)



# Routing solution

On a routing digraph, at each node (but Target),
let choose (double lines) a unique outgoing arc.
$\Rightarrow$ From each node, this induces a unique path/word.

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- The induced path leads to Target. (No looping induced path.)

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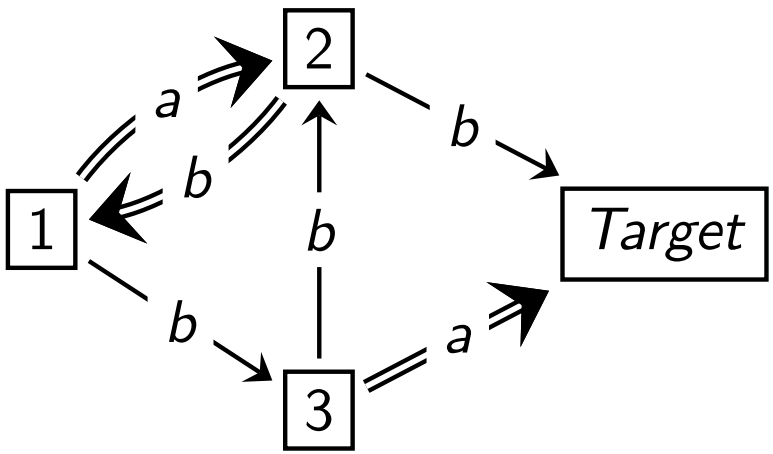
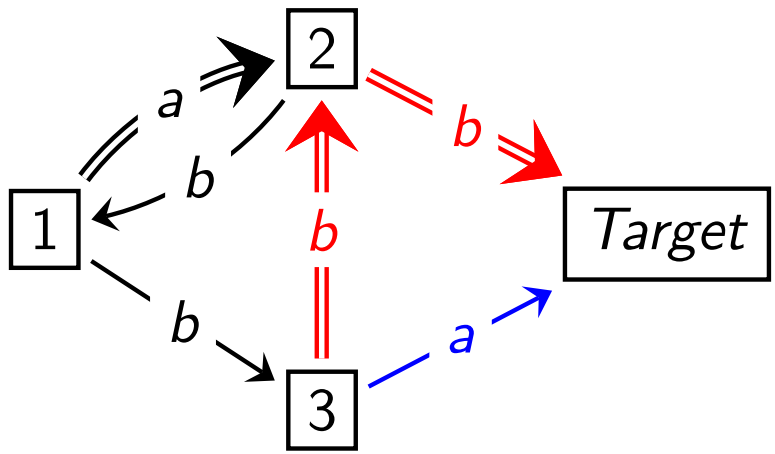
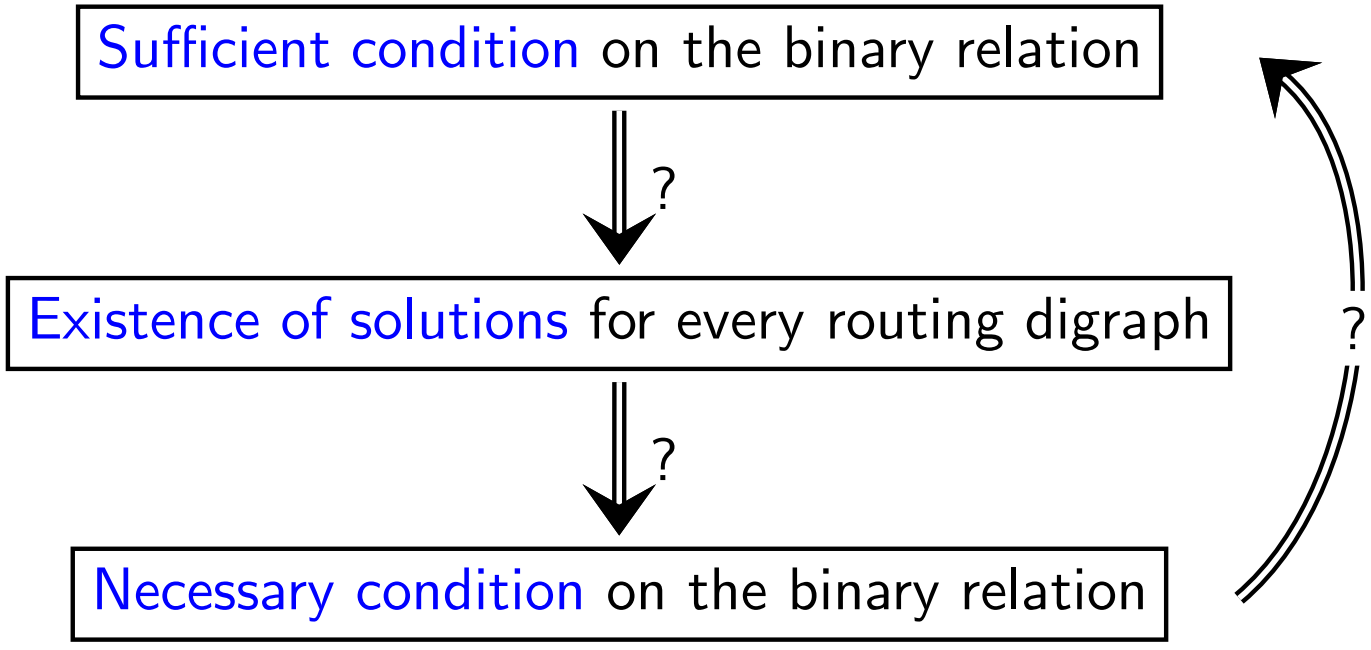



This is a routing solution if from each node:

- The induced path leads to Target. (No looping induced path.)
- And it is maximal *w.r.t* the binary relation. (But $bb \prec a$.)

# Issues




Routing problems are not uniform:

- Target: no outgoing arc but reachable from everywhere.
- Induced path: finite when leading to Target; infinite otherwise.

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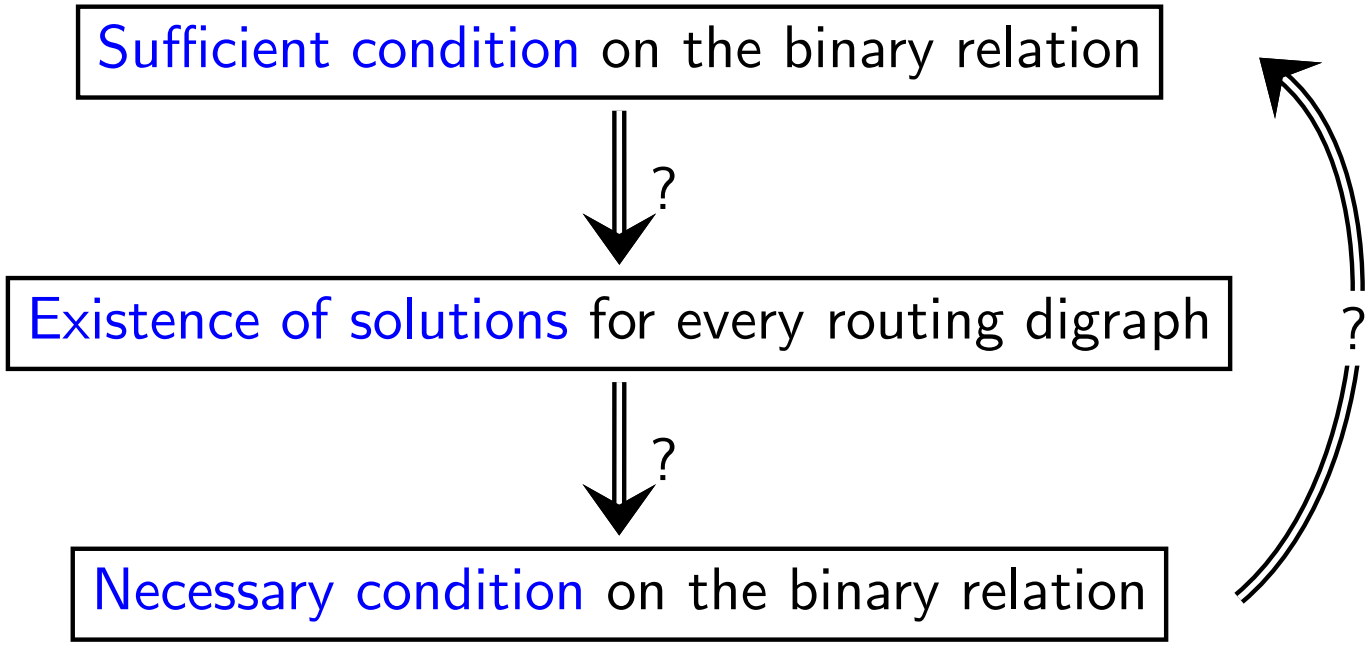


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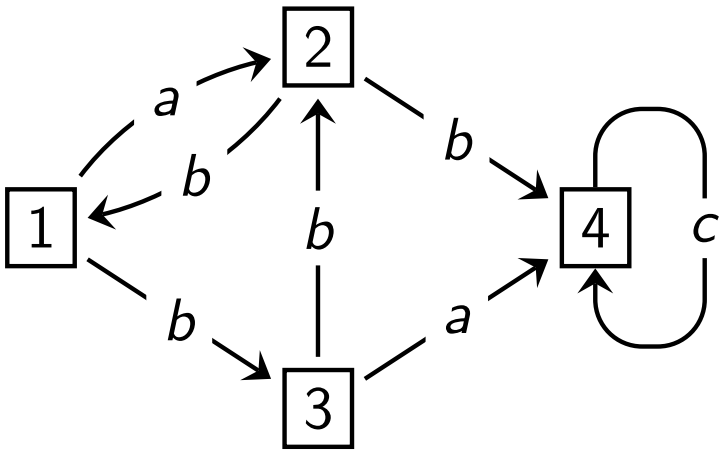
# Dalograph and strategy

Dalograph: Directed, arc-labelled graph with non-zero outdegree.



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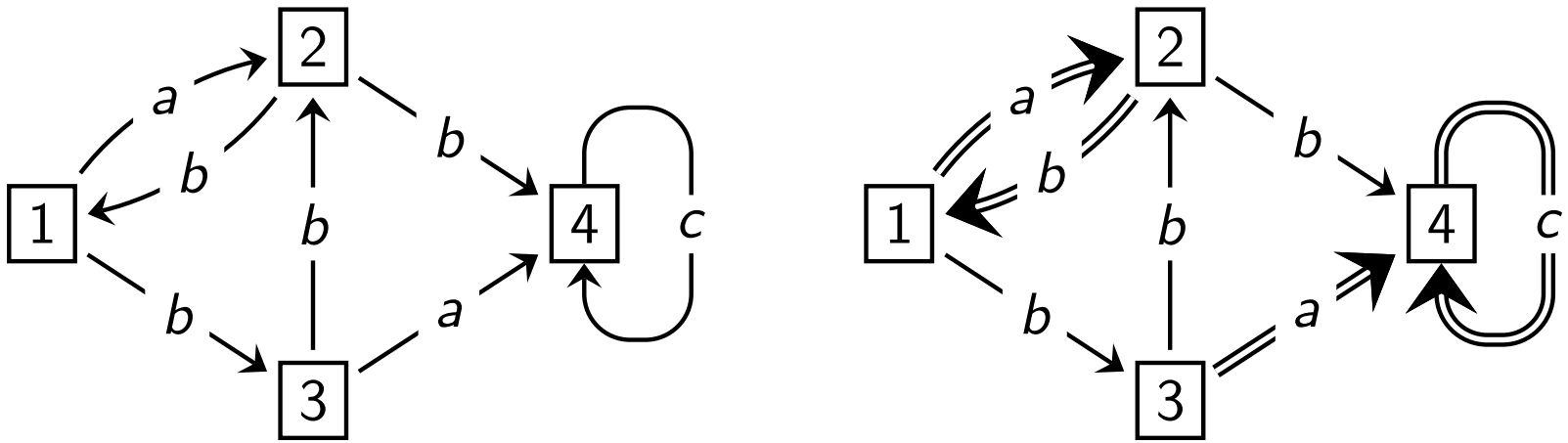


A strategy amounts to choosing a unique outgoing arc per node.

- Each node induces a unique ultimately-periodic path/word.
- Node $\boxed{1}$ induces $(ab)^\omega = ababababab\ldots$
- Node $\boxed{3}$ induces $ac^\omega = acccccc\ldots$

# Preference and equilibrium

**Preference:** binary relation over ultimately-periodic words.
*e.g.* let define $\prec$ by $(ba)^\omega \prec bc^\omega$ and $bbc^\omega \prec ac^\omega$.

**Equilibrium:** strategy inducing at each node
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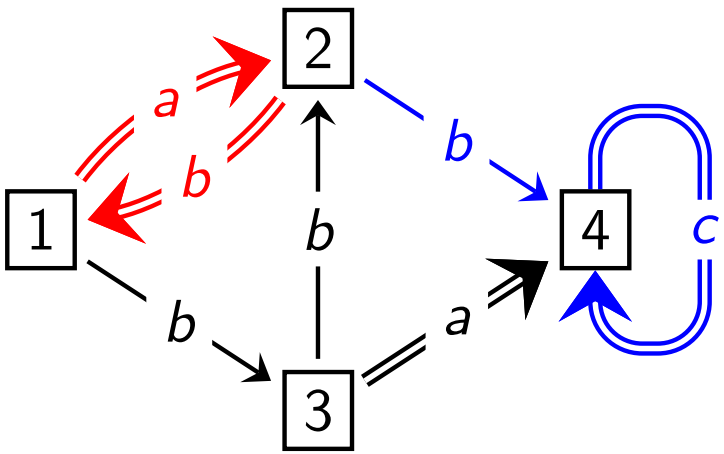

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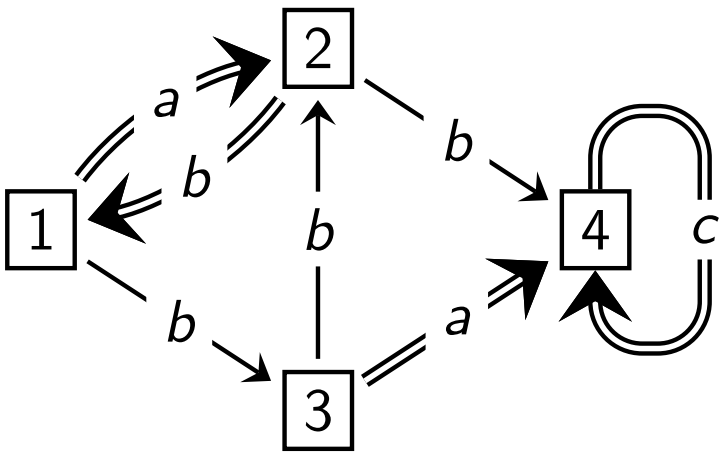


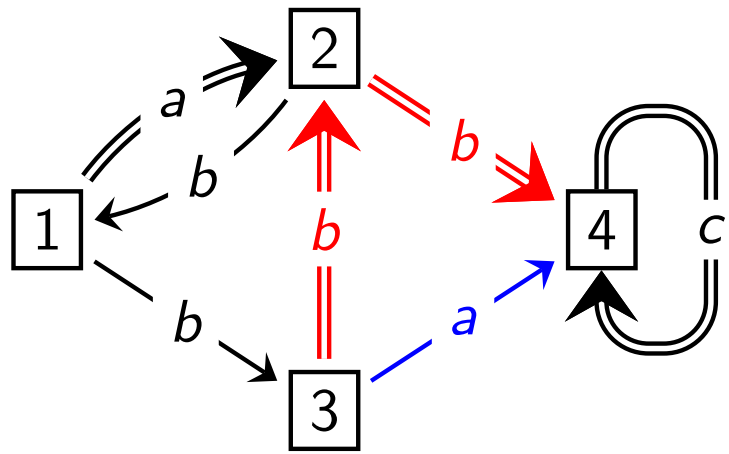

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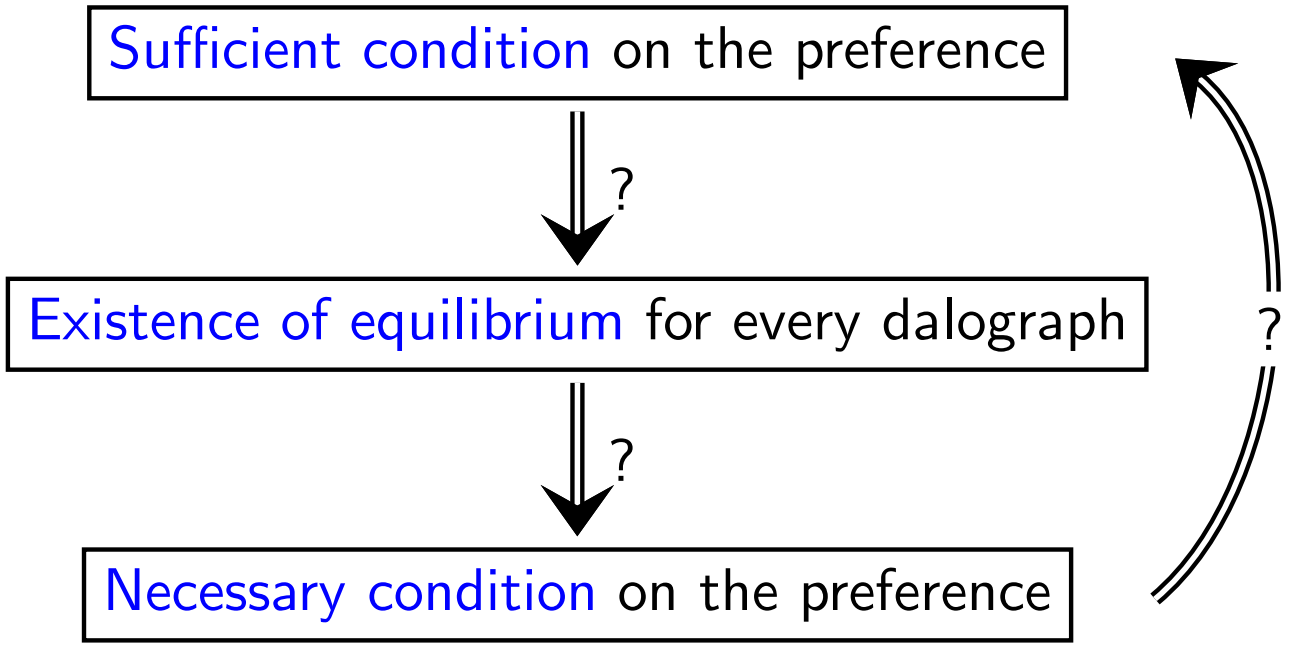


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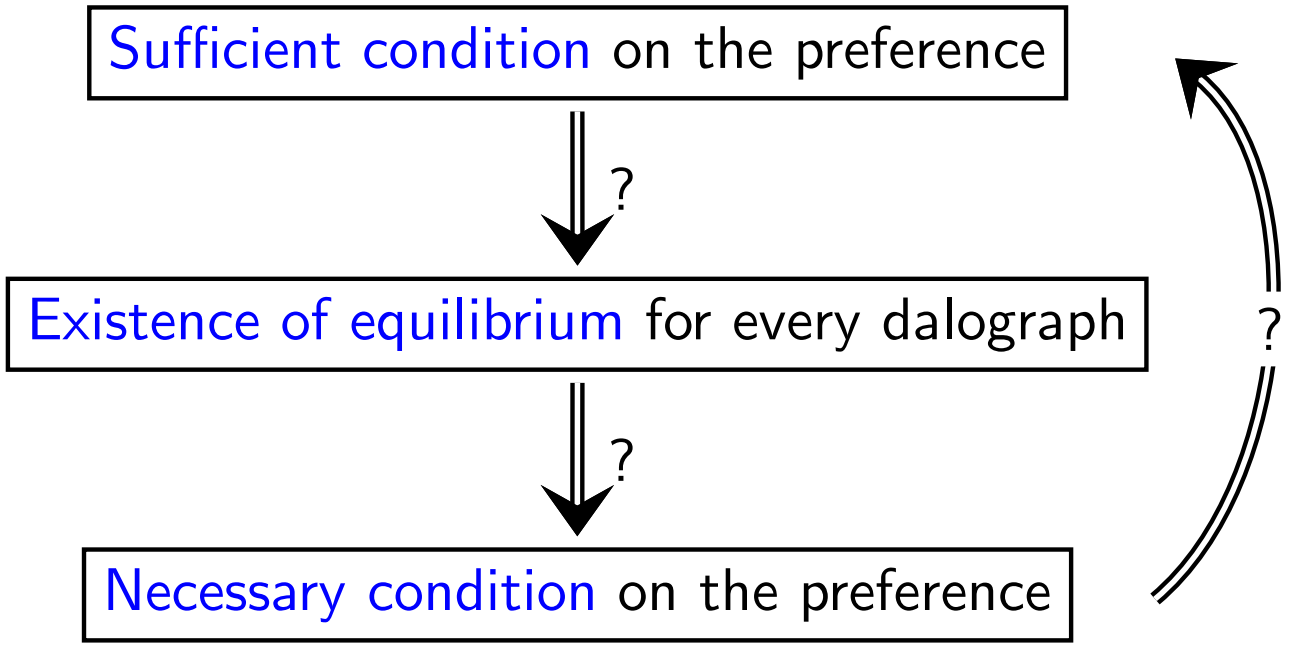


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Dalograph: existence of solutions
- Sufficient condition on the preference
- Necessary condition on the preference

Routing: existence of solutions

# Binary relations: 3 definitions

## Definition (Strict weak order, exists in the literature)

*Asymmetric relation whose negation is transitive.*

$x \prec y \Rightarrow y \not\prec x$ *and* $x \not\prec y \wedge y \not\prec z \Rightarrow x \not\prec z$

*(Between strict partial order and strict linear order.)*

## Definition (E-prefix)

$u\alpha \prec u\beta \Rightarrow \alpha \prec \beta$ *Preservation by common prefix elimination.*

## Definition (Subcontinuity)

$\alpha \prec u\alpha \Rightarrow u^\omega \not\prec \alpha$

*Improving prefix insertion implies harmless prefix repetition.*

## Example (Comparing infinite words *via* their initials)

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*Maximal path whose first proper subpath is hereditary maximal.*

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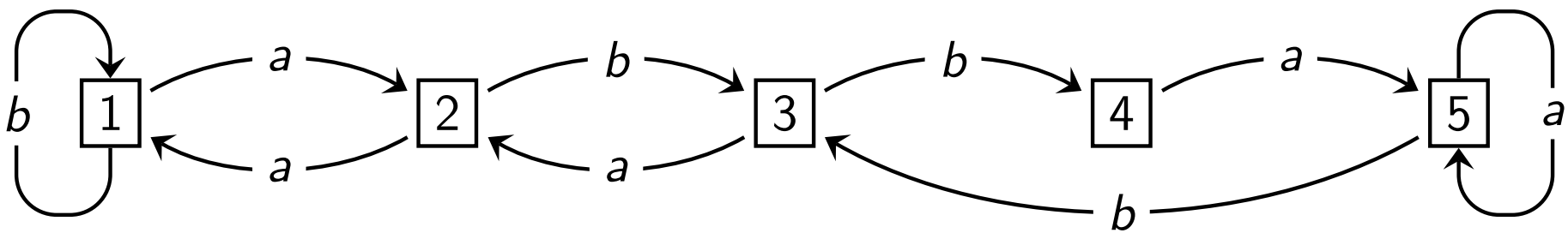

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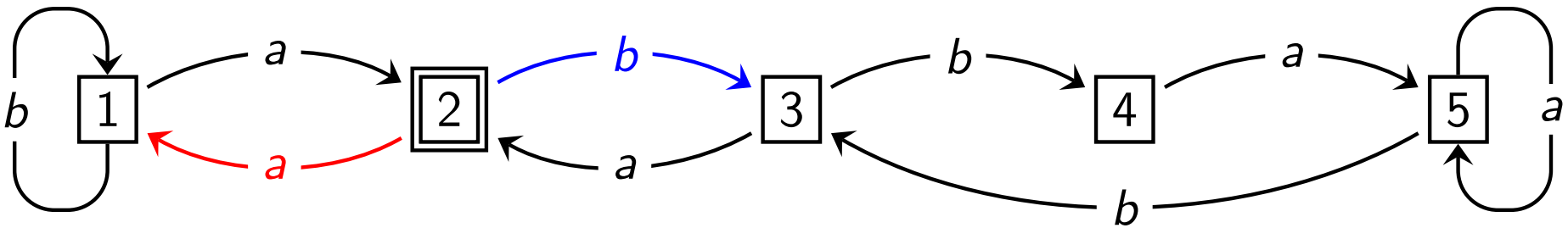
**Consider maximality at node** $2$**.**



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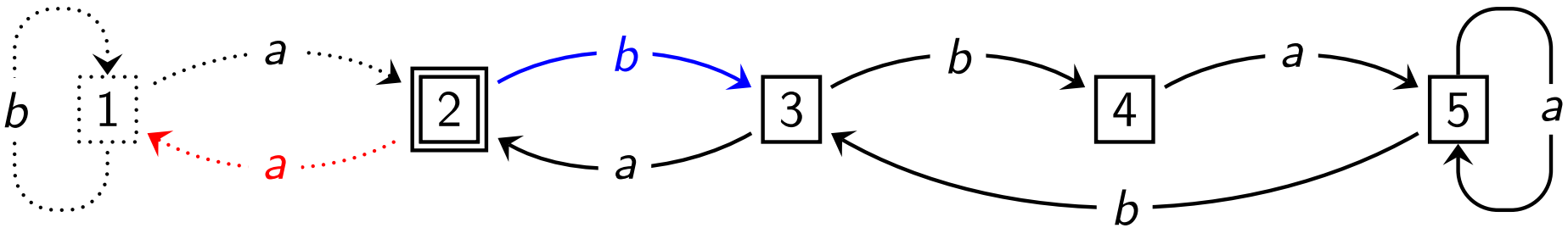
**Discard the pathwise "useless" part of the dalograph.**



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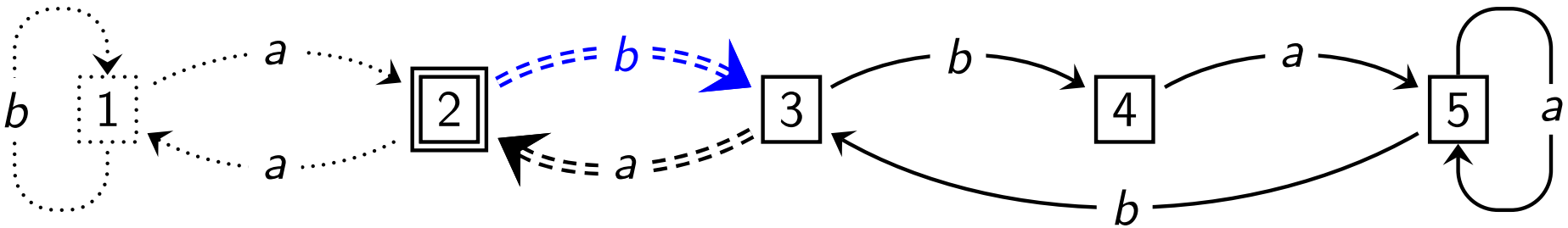
**Pick one maximal path at node** 2**.**



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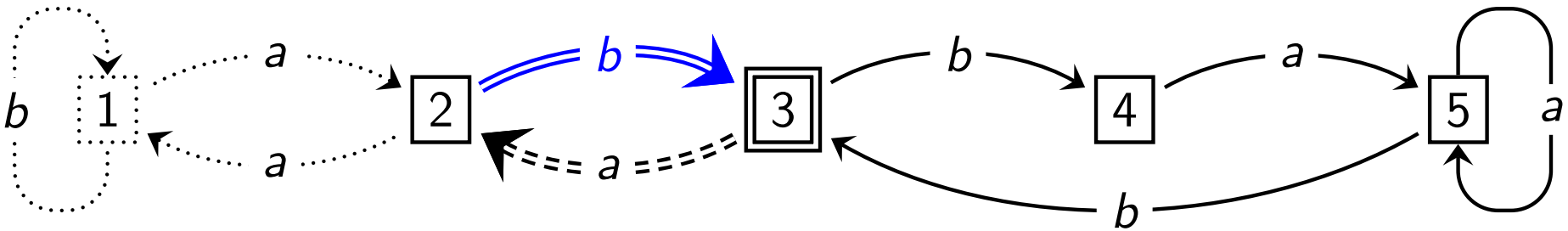
**Move one step along the chosen path.**



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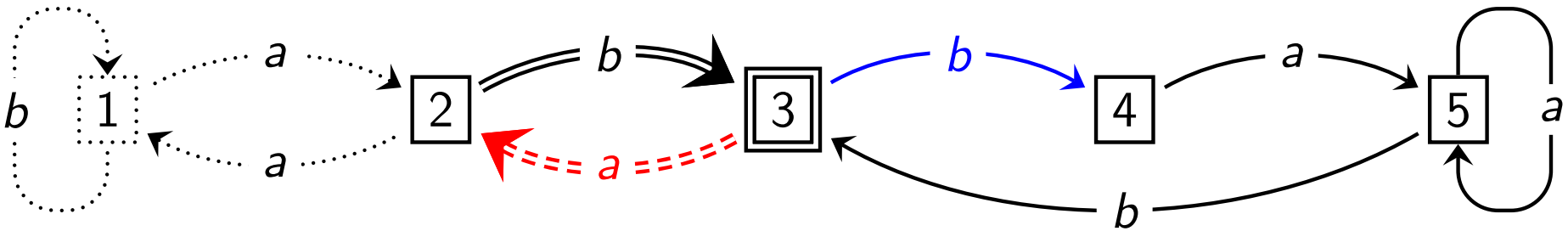
**Consider maximality at node** 3**.**



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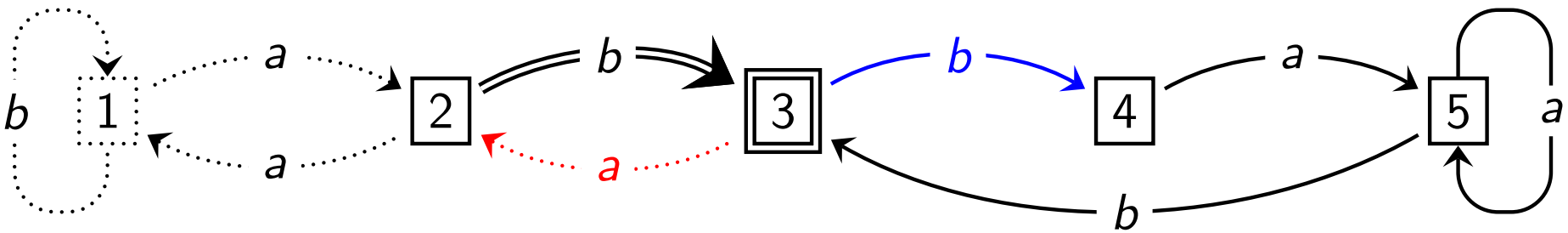
**Discard the pathwise "useless" part of the dalograph.**
**This discards the chosen subpath from node** 3.



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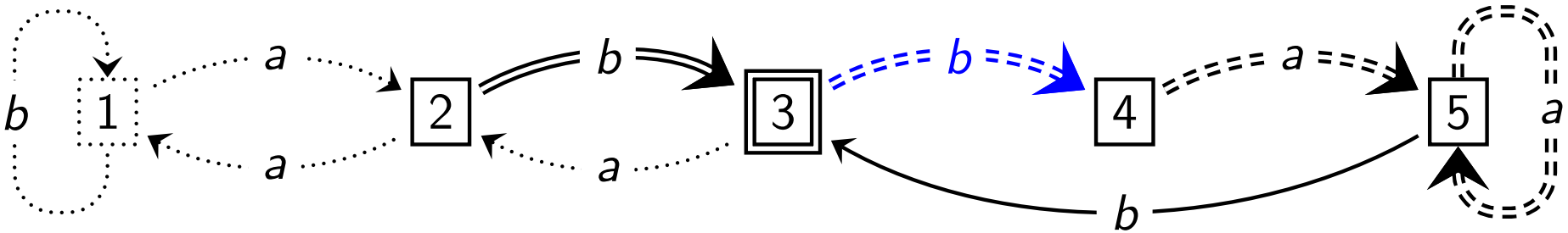

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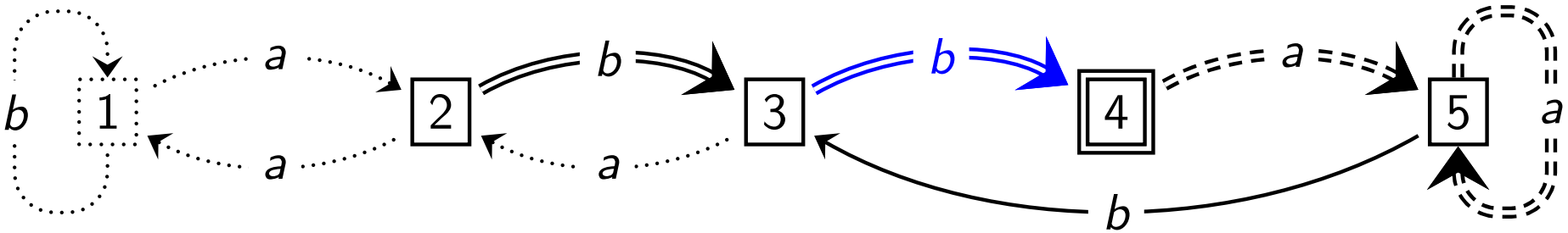
**Move one step along the new chosen subpath.**



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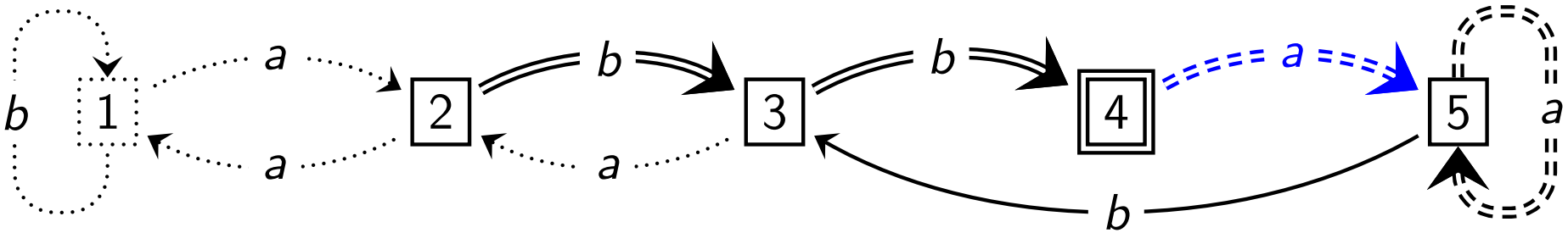
**The subpath starting from node** 4 **is still maximal.**



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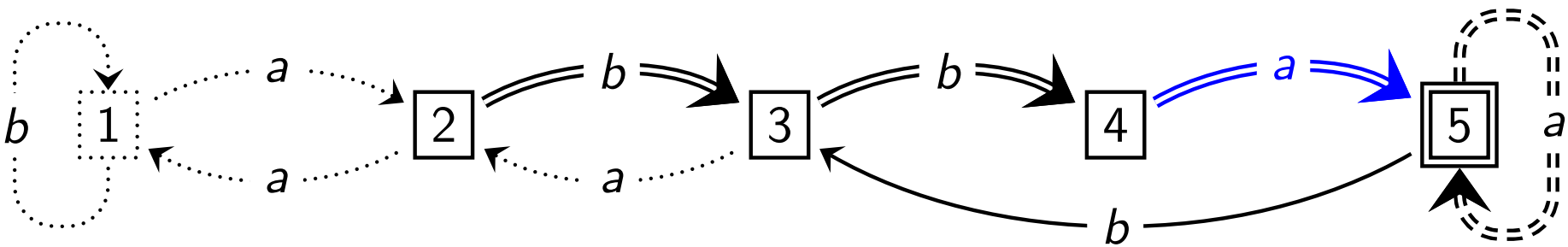

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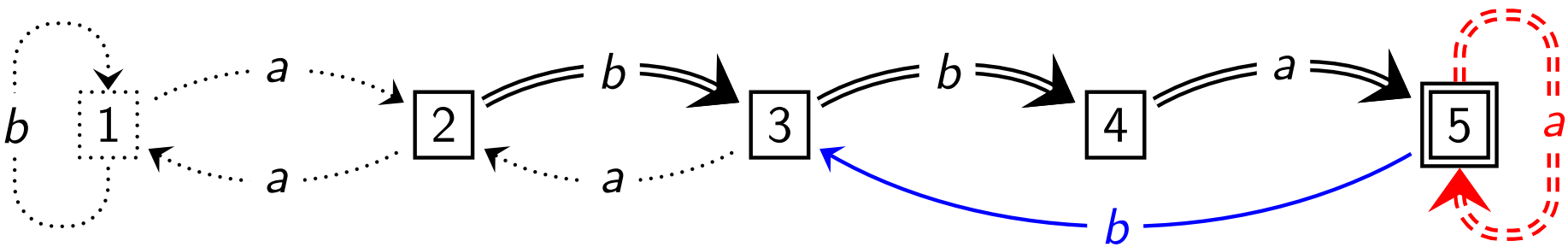

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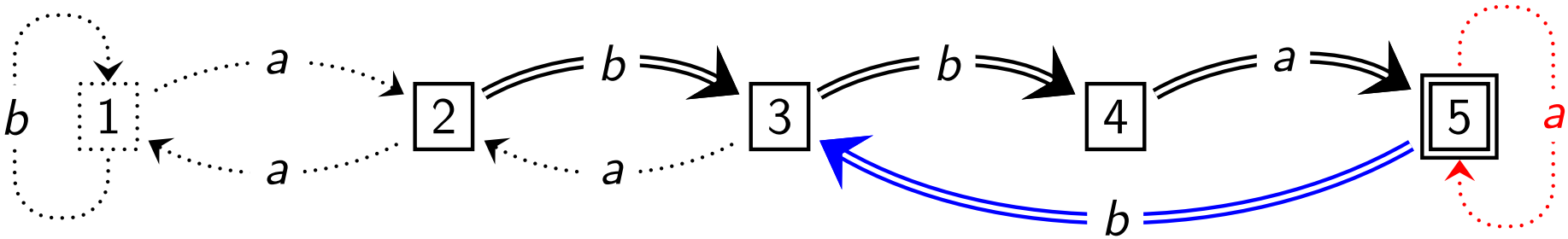
*Let* $\Sigma = \{a, b\}$ *and* $a \cdot \Sigma^\omega \prec b \cdot \Sigma^\omega$.

**Discard the pathwise "useless" part of the dalograph.**
**$\Rightarrow$ One sole possibility to complete the looping path.**



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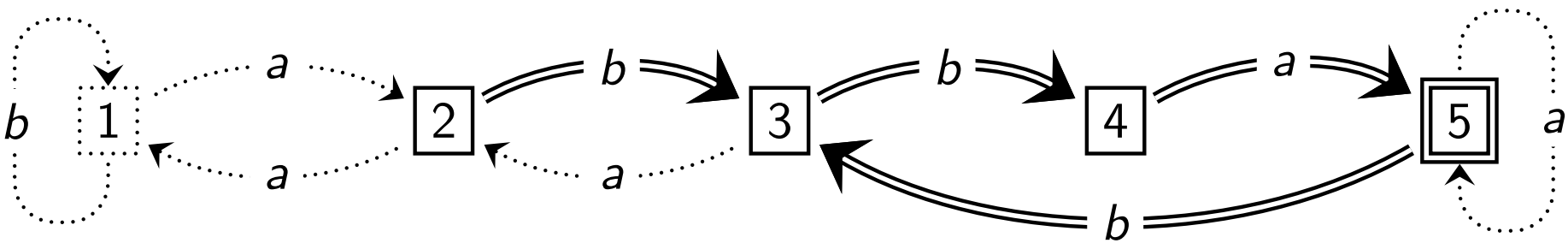


## Theorem

*If the preference is included in a E-prefix and subcontinuous strict weak order, then every path built as above is hereditary maximal.*

# Sufficient condition

## Theorem

*If the preference $\prec$ is included in a E-prefix and subcontinuous strict weak order, then every dalograph has an equilibrium.*

## Proof (scetch):

*Preliminary remarks:*

- wlog, $\prec$ is a E-prefix and subcontinuous strict weak order.
- If each node of a dalograph has one sole outgoing arc, then there is one sole possible strategy $\Rightarrow$ equilibrium.

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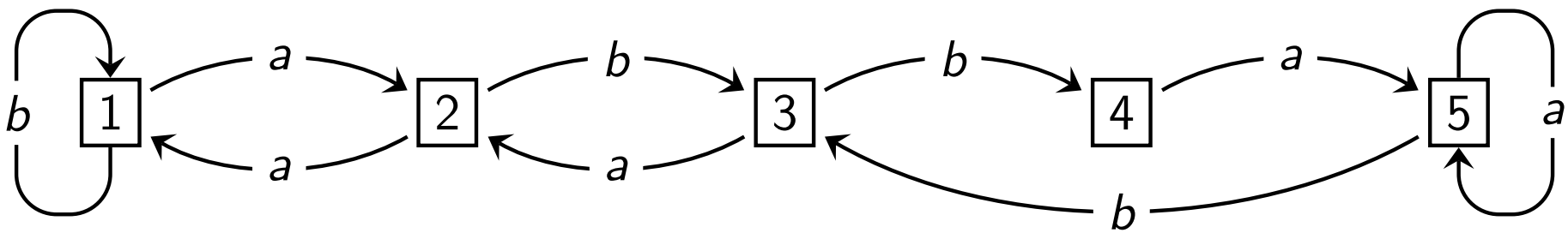
**Proof (scetch):**

By induction on the number of arcs (easy base case).

In the example below, let $\Sigma = \{a, b\}$ and $a \cdot \Sigma^\omega \prec b \cdot \Sigma^\omega$.



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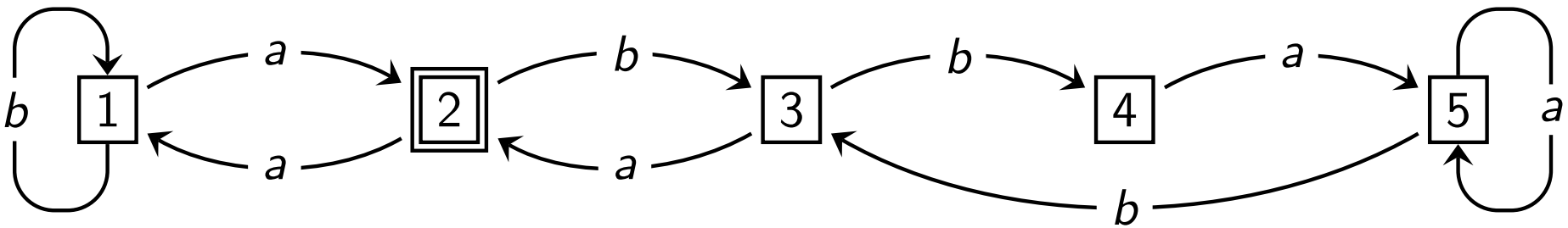
**Let node 2 have at least two outgoing arcs.**
(If there is no such node, then the only strategy is an equilibrium.)



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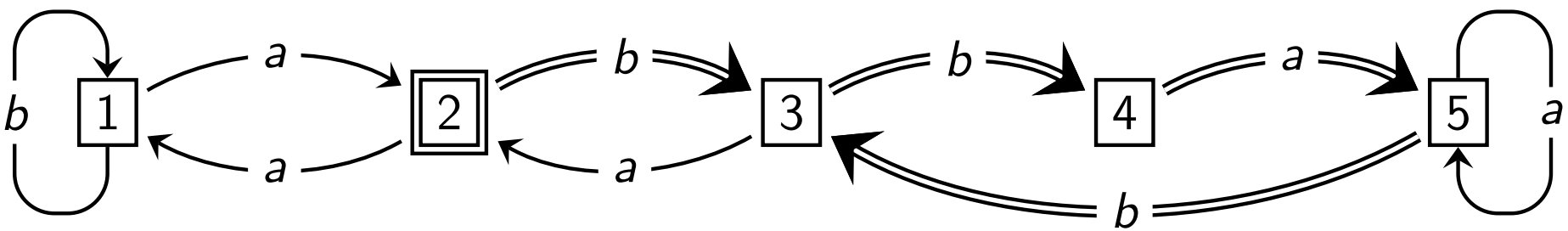
**Compute a hereditary maximal path starting from node 2.**
(This is possible by the assumptions and the previous theorem.)



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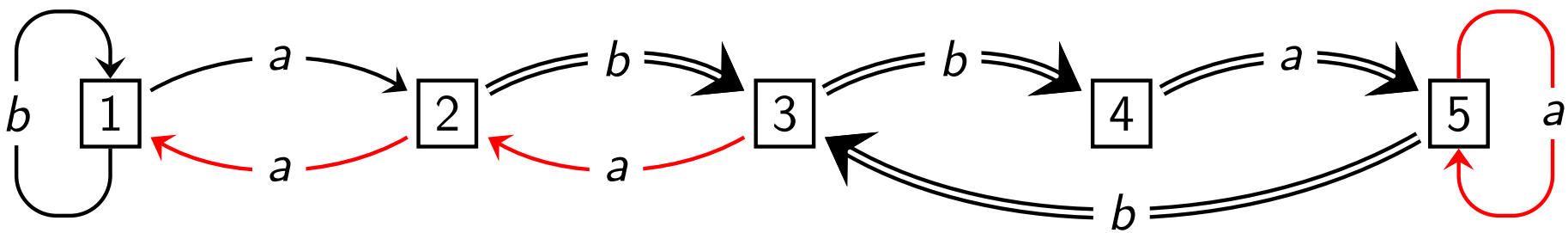
**Consider the arcs starting on this path, but not in this path.**



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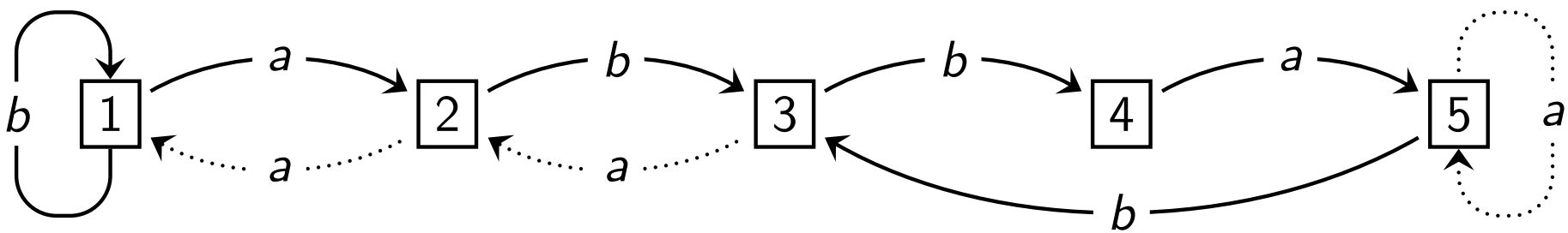
In the example below, let $\Sigma = \{a, b\}$ and $a \cdot \Sigma^\omega \prec b \cdot \Sigma^\omega$.

**Remove these arcs and get a smaller dalograph.**



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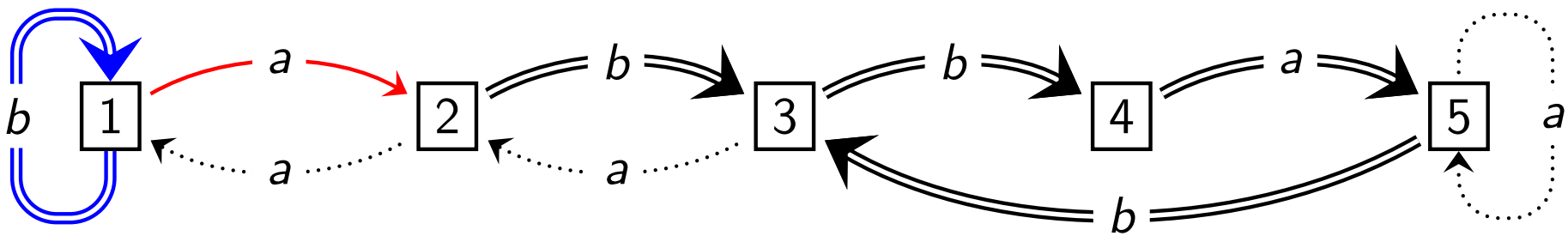
**By I-H, there is an equilibrium for the smaller dalograph.**
(Which involves the above-mentionned hereditary maximal path.)



# Sufficient condition

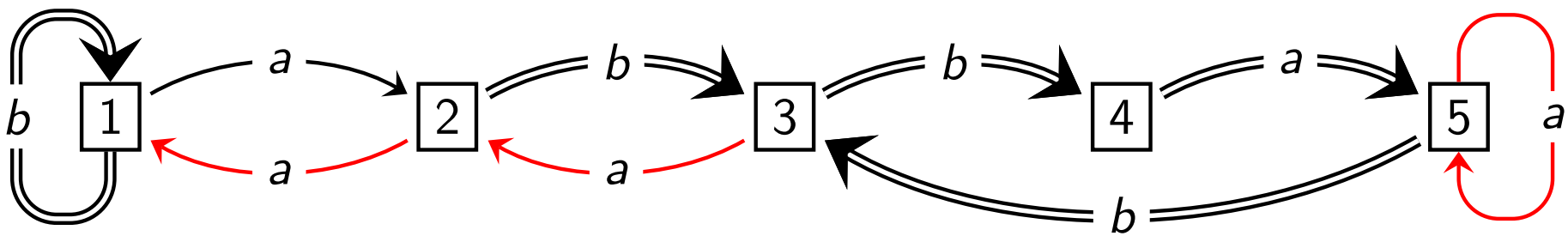
**Theorem**

*If the preference $\prec$ is included in a E-prefix and subcontinuous strict weak order, then every dalograph has an equilibrium.*

**Proof (scetch):**

By induction on the number of arcs (easy base case).
In the example below, let $\Sigma = \{a, b\}$ and $a \cdot \Sigma^\omega \prec b \cdot \Sigma^\omega$.

**Put back the arcs that were removed earlier.**



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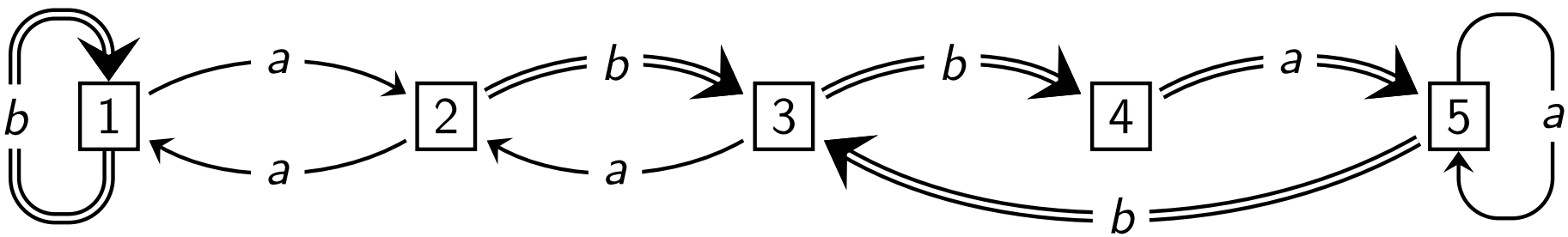
**This is an equilibrium, by the assumptions.** End of proof.



# Outline

Routing: definitions

Dalograph: definitions

Dalograph: existence of solutions

- Sufficient condition on the preference
- Necessary condition on the preference

Routing: existence of solutions

# Equilibrium preservation by E-prefix closure

E-prefix closure: $\alpha \prec^{ep} \beta \quad \stackrel{\Delta}{=} \quad \exists u,\ u\alpha \prec u\beta.$

Example: $(ab)^\omega \prec a^\omega \quad \Rightarrow \quad (ab)^\omega \prec^{ep} a^\omega \quad \wedge \quad (ba)^\omega \prec^{ep} a^\omega$

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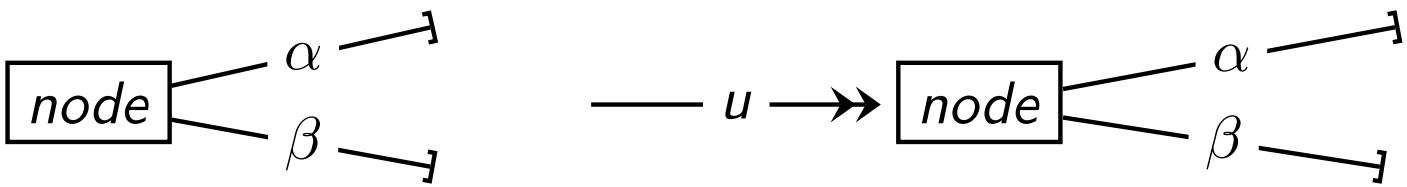
## Lemma

*All dalographs have $\prec$-equilibria $\Rightarrow$ they have $\prec^{ep}$-equilibria*

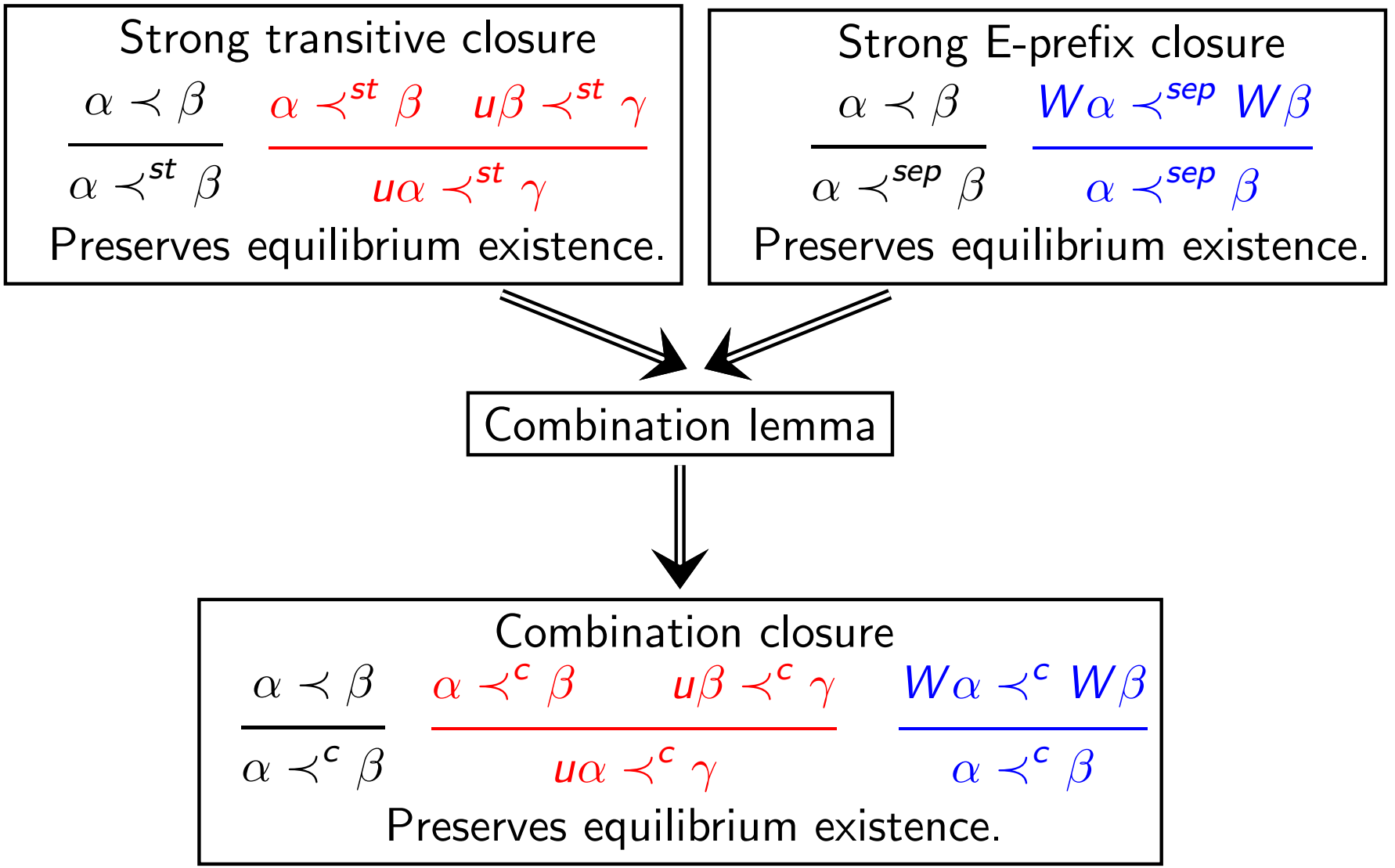
## Proof.

Let $g$ be a dalograph. At all nodes with eligible words $\alpha$ and $\beta$ such that $\alpha \prec^{ep} \beta$, add an incoming $u$ such that $u\alpha \prec u\beta$.

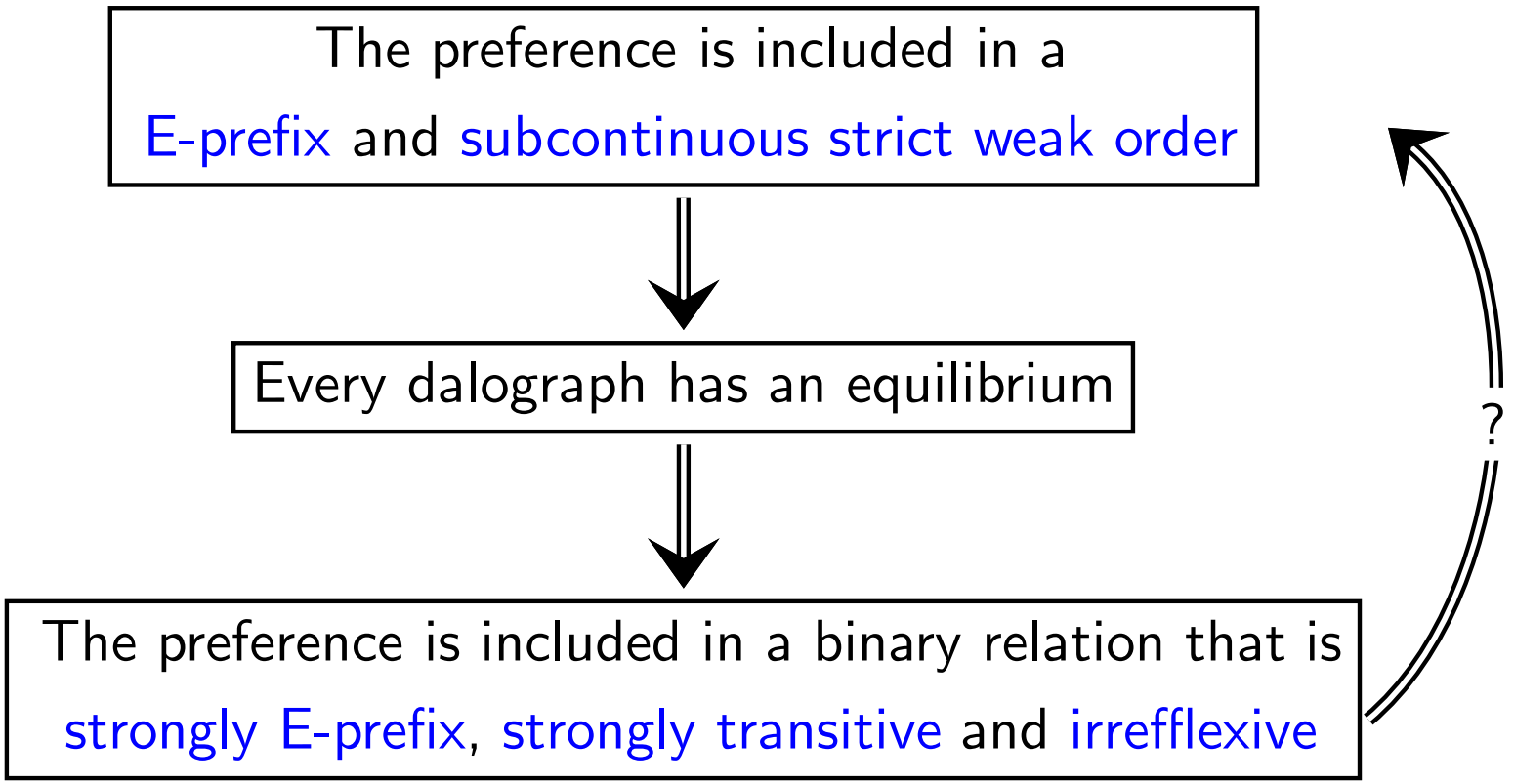



This new dalograph $g_1$ has a $\prec$-equilibrium. For such $\alpha$'s, $\beta$'s and $u$'s, the equilibrium chooses $u\beta$ over $u\alpha$, so it chooses $\beta$ over $\alpha$. Removing the $u$'s yields a $\prec^{ep}$-equilibrium for $g$. □

# Combination of equilibrium-preserving closures



# Summary




**Theorem**

*If a preference is a strict linear order:*

*It is E-prefix and subcontinuous $\Leftrightarrow$ all dalographs have equilibria*

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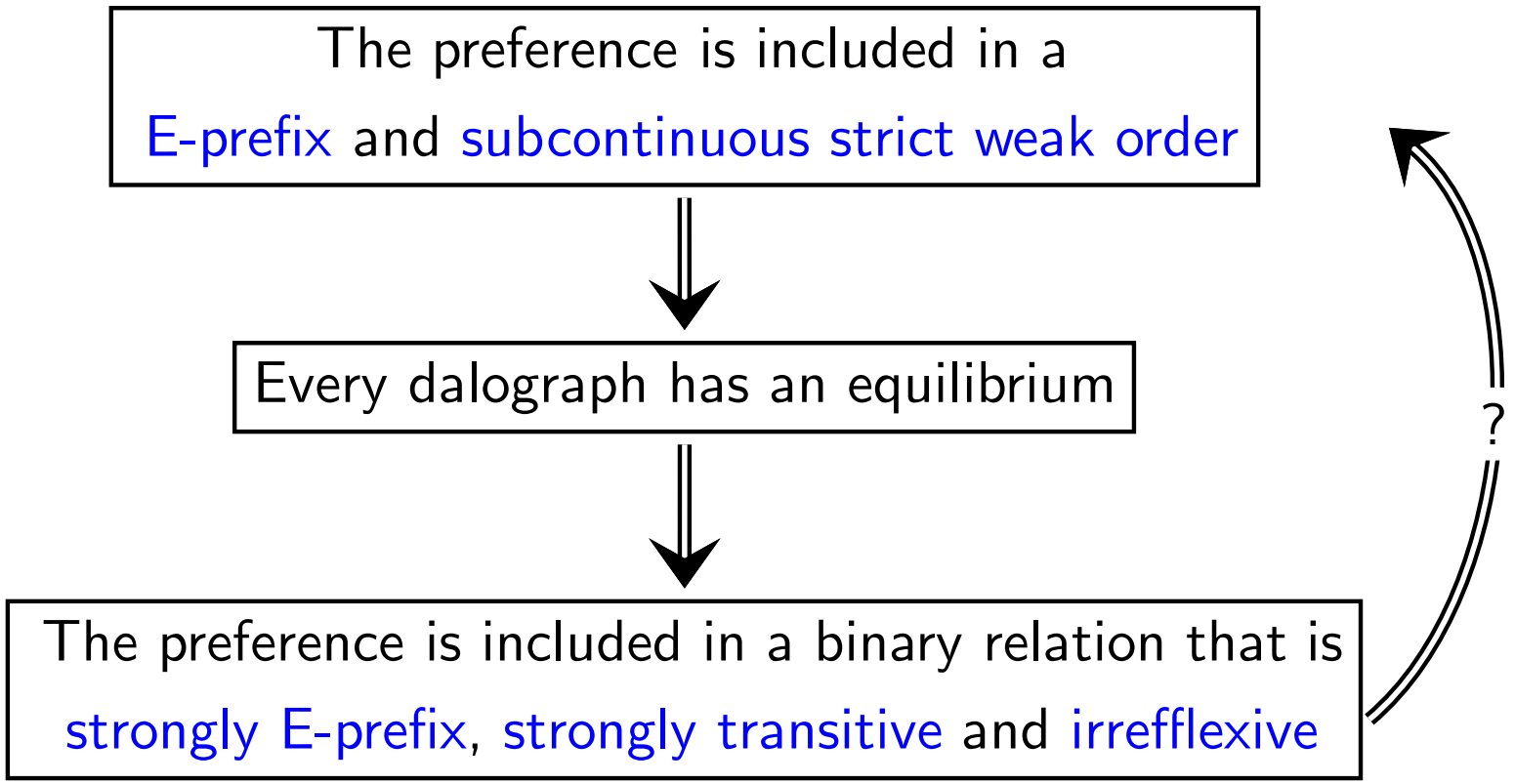


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# Outline

Routing: definitions

Dalograph: definitions

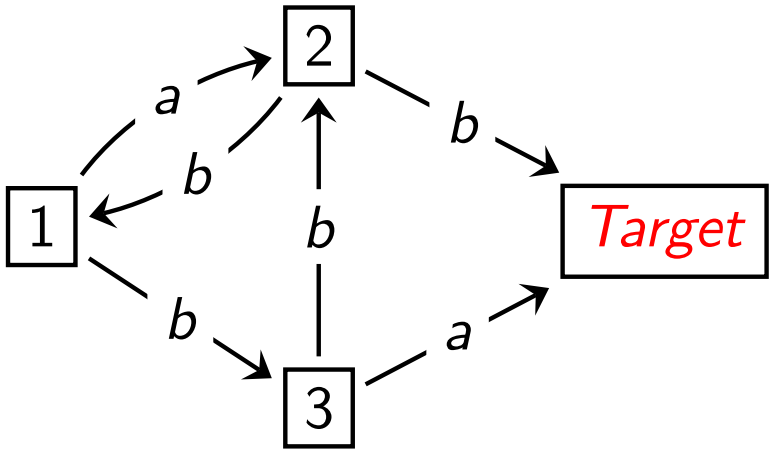
Dalograph: existence of solutions

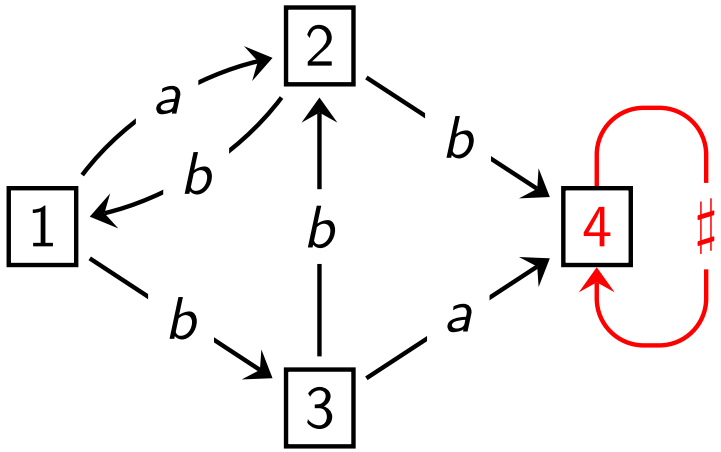
Routing: existence of solutions

# Translation from routing to dalograph

$$\Sigma \mapsto \Sigma \cup \{\sharp\}$$

$$\frac{u \prec_r v}{u\sharp^\omega \prec_d v\sharp^\omega} \qquad \frac{\sharp \notin \alpha}{\alpha \prec_d u\sharp^\omega}$$




**Theorem**

*A preference is included in an E-prefix strict weak order $\prec$ such that $v \not\prec uv \Rightarrow$ every routing problem has a solution.*

**Theorem (For strict linear order preference)**

*A preference is E-prefix and $uv \prec v$ ($u$ non-empty)*
$\Leftrightarrow$ *Every routing problem has a solution.*

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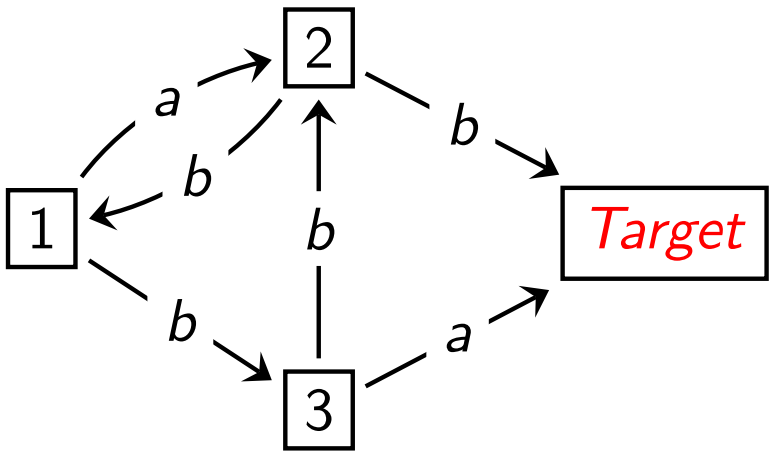


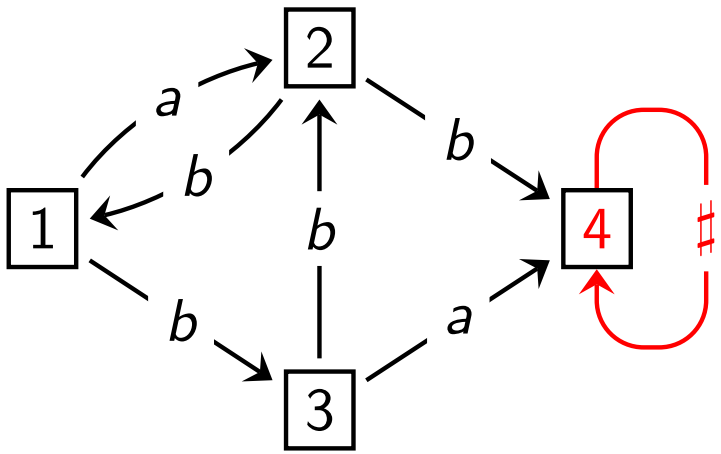


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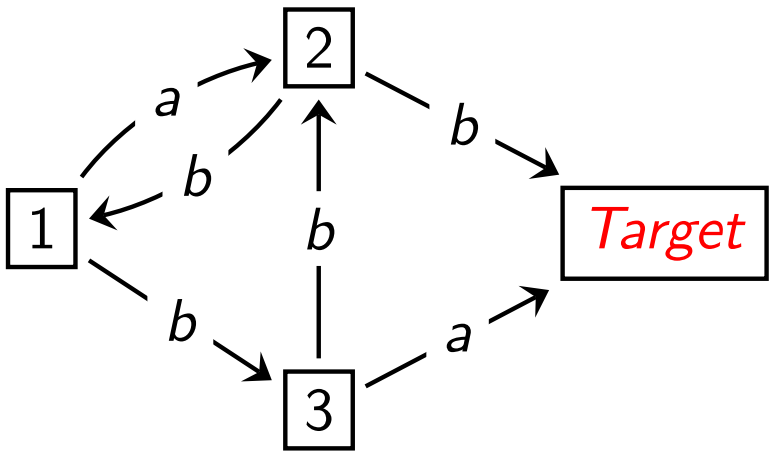


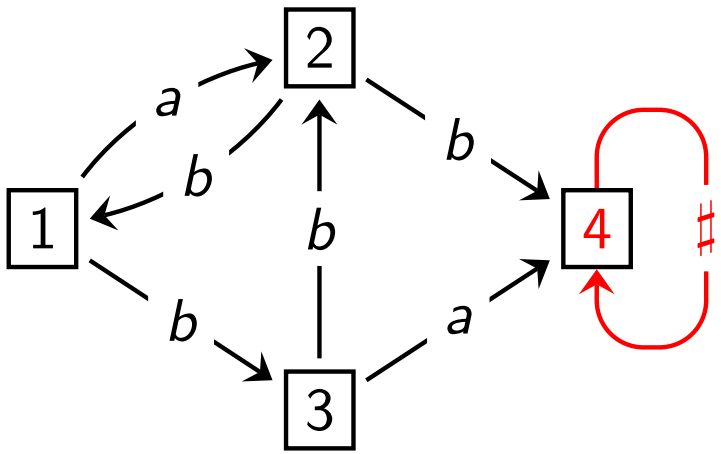


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# Conclusion

## Summary:

- Definitions of dalograph and equilibrium.
- Sufficient condition for equilibrium existence.
  (hereditary maximal path + induction)
- Necessary condition for equilibrium existence.
  (preservation by closure + combination lemma)
- Constructive proofs $\Rightarrow$ building equilibria or counter-examples.
- Application to routing issues.

## Possible continuations:

- Dalographs: seek a full equivalence property.
- Apply dalographs to other routing problems or other fields