Corinne Touati, INRIA

Robot Cockroach Tests Insect Decision-Making Behavior
Optimality of Multi-user system

- Optimality of a *single* user

![Graph showing utility vs. parameter with an optimal point marked]

Utility Optimal point Parameter

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Situation with multiple users

Utility of user 2

Utility of user 1

Good for user 2

Good for user 1

Corinne Touati (INRIA)
Introduction to Game Theory
Optimality of Multi-user system

Definition.

A point is Pareto optimal if it cannot be strictly dominated.
Optimality of Multi-user system

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<th>Non-cooperative games</th>
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<tr>
<td>Institution setting rules</td>
<td>Individual behavior</td>
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<tr>
<td>and penalties to enforce them</td>
<td>converge (or not) to an equilibrium</td>
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Example: Routing intersection:

- **Cooperative approach**: set of road signs (traffic lights, “stop signs”...) enforced by the police
- **Non-cooperative approach**: everyone tries to cross it as quickly as possible
Outline

1 Non-cooperative optimization
   - Nash Equilibria
   - Braess Paradoxes
   - Dynamic games
   - Other equilibria

2 Cooperative Games
   - Definitions of fairness
   - Examples
   - Non-convex systems

3 Other yet interesting topics...
Outline

1. Non-cooperative optimization
   - Nash Equilibria
   - Braess Paradoxes
   - Dynamic games
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3. Other yet interesting topics...
Nash equilibria: definition

Definition
In a non-cooperative setting, each player makes a decision so as to maximize its own return.

Nash equilibria
In a Nash equilibrium, no player has incentive to unilaterally modify his strategy.

\[ s^* \text{ is a Nash equilibrium iff:} \]
\[ \forall p, \forall s_p, u_p(s_1^*, \ldots, s_p^*, \ldots, s_n^*) \geq u_p(s_1^*, \ldots, s_p, \ldots, s_n^*) \]
Pros
▶ Intuitive

Cons
### Nash Equilibria: definition (cont.)

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Nash equilibria: Applications

Various contexts:

- **Load balancing systems**
  Users decide which server to send their request so as to minimize their average delay.

- **Wireless systems**
  Users decide what power to use so as to maximize a compromise between the transfer rate and the battery usage.

- **Pricing systems**
  Providers choose their prices so as to maximize their revenue, which is a function of their charged price and their infrastructure cost and market share.

- **Queuing systems**
  Users optimize their "power" defined as the ratio of their throughput and their expected delay.
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Nash equilibria: Application to scheduling of bg-of-task applications

Two computers / two applications

\[ b_1 = 1 \]
\[ b_2 = 2 \]

\[ w_1 = 2 \]
\[ w_2 = 1 \]
Nash equilibria: Application to scheduling of bg-of-task applications

Two computers / two applications

Cooperative Approach:
Application $i$ is processed exclusively on computer $i$.
$\alpha_{i}^{(coop)} = \alpha_{2}^{(coop)} = 1.$
Nash equilibria: Application to scheduling of bg-of-task applications

Two computers / two applications

Cooperative Approach:
Application $i$ is processed exclusively on computer $i$.  
$\alpha_{1}^{(coop)} = \alpha_{2}^{(coop)} = 1$.

Non-Cooperative Approach: 
$\alpha_{1}^{(nc)} = \alpha_{2}^{(nc)} = \frac{3}{4}$
Hypothesis: packets select their routes of travel from an origin to a destination so as to minimize their own travel cost.

- 3 possible routes
- cost of links are proportional to the fraction of users $x$ passing through it.

Difference with the previous example?
Hypothesis: packets select their routes of travel from an origin to a destination so as to minimize their own travel cost.

- The number of users is infinite
- Each of them has a negligible impact

Belongs to the class of “population games”
### Population games

**Definition:** Population game.

- $Q$ non atomic populations, each of them of mass $\hat{d}_q$
- A finite set of strategies for each population
- A strategy distribution $y = (y_1, ..., y_Q)$, where $y_q$ is a vector containing the masses of the subset of population $q$ adopting each possible strategy
- The marginal payoff per unit of class $i$ of population $q$: $F^i_q(y)$

**Definition:** Wardrop equilibrium.

A state $\hat{y}$ is a Wardrop equilibrium if, for any population:

- All strategies being used by members of the population yelf the same marginal payoff: $\forall i, j, y^i_q \neq 0, y^j_q \neq 0, F^i_q(\hat{y}) = F^j_q(\hat{y})$
- The marginal payoff associated to all strategies actually used by members is lower than it would be with any of the strategies not chosen.
Wardrop equilibria: application

- 1 population \((Q = 1)\), 3 possible strategies
- Strategy distribution \(y = (y_1)\) with \(y_1 = (m_1, m_2, m_3)\)
- Marginal payoff per unit:

\[
\begin{align*}
F_1^1(y) &= 10 \times (m_1 + m_3) + (m_1 + 50) \\
&= 11 \cdot m_1 + 10 \cdot m_3 + 50 \\
F_1^2(y) &= (m_2 + 50) + 10 \times (m_2 + m_3) \\
&= 11 \cdot m_2 + 10 \cdot m_3 + 50 \\
F_1^3(y) &= 10 \times (m_1 + m_3) + (m_2 + 10) + 10 \times (m_2 + m_3) \\
&= 10 \cdot m_1 + 20 \cdot m_3 + 11 \cdot m_2 + 10
\end{align*}
\]
Let $\hat{y}$ be the strategy distribution at the Wardrop equilibria. Then,

$$\forall i, j, m_i \neq 0, m_j \neq 0, F^i_1(y) = F^j_1(y),$$

and

$$\forall i, j, m_i \neq 0, m_j = 0, F^i_1(y) < F^j_1(y).$$
Wardrop equilibria: application

Total population mass
\[ m_1 + m_2 + m_3 = 6 \] and:
\[
F_1^1(y) = 11m_1 + 10m_3 + 50 \\
F_1^2(y) = 11m_2 + 10m_3 + 50 \\
F_1^3(y) = 10m_1 + 11m_2 + 20m_3 + 10
\]

Suppose only routes “a” and “b” are used \((m_3 = 0)\), then
\[ m_1 = 3 \] and \( F_1^1(y) = F_1^2(y) = 83 \).

But the single cost of a packet going through path “c” would be
\[ 10m_1 + 11m_2 + 10 = 73 < F_1^1(y), \]

hence \((m_1m_2 \neq 0) \Rightarrow m_3 \neq 0\).
Total population mass
\[ m_1 + m_2 + m_3 = 6 \] and:
\[ F^1_1(y) = 11.m_1 + 10.m_3 + 50 \]
\[ F^2_1(y) = 11.m_2 + 10.m_3 + 50 \]
\[ F^3_1(y) = 10.m_1 + 11.m_2 + 20.m_3 + 10 \]

With similar arguments, we can show that \( m_1.m_2.m_3 \neq 0 \).
Hence \( F^1_1(y) = F^2_1(y) = F^3_1(y) \).
Then \( m_1 = m_2 \)
and \( 11.m_1 + 10.m_3 + 50 = 21.m_1 + 20.m_3 + 10 \),
hence \( 40 = 10.m_3 + 10m_1 \).
Finally \( m_1 = m_2 = m_3 = 2 \) and \( F^1_1(\hat{y}) = F^2_1(\hat{y}) = F^3_1(\hat{y}) = 92 \).
There is actually a simpler way :)  

### Potential games

Here, potential function $\Phi(m_1, m_2, m_3) = \sum_{l \text{ links}} \int_0^{\alpha_l} c_l(u)du$

with $\alpha_l = \sum_{p \text{ paths}} m_l \delta_{l,p}$,  

$\delta_{l,p} = \begin{cases} 
1 & \text{if flow } l \text{ goes through path } p \\
0 & \text{otherwise}
\end{cases}$

and $c_l$ the cost of crossing link $l$.

Then the Wardrop equilibria is the solution of:

$\hat{m} = (\hat{m}_1, \hat{m}_2, \hat{m}_3), \ \text{argmin } \Phi(m) \ \text{subject to } \sum m_i = 6.$
Important remark

We saw that \( F_1^1(\hat{y}) = F_1^2(\hat{y}) = F_1^3(\hat{y}) = 92 \).

But also that, if only routes “a” and “b” were used (\( m_3 = 0 \)), then

\[ m_1 = 3 \text{ and } F_1^1(y) = F_1^2(y) = 83. \]

(But the cost of a single packet going through path “c” would be \( 73 < F_1^1(y) \)).
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2. Cooperative Games
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   - Examples
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3. Other yet interesting topics...
Context: urban transportation networks.
Hypothesis: travelers select their routes of travel from an origin to a destination so as to minimize their own travel cost or travel time.

Rate: 6

With 2 roads,
\[ \text{Cost}_a = \text{Cost}_b = 83 \]
Braess Paradoxes: definition

Context: urban transportation networks.
Hypothesis: travelers select their routes of travel from an origin to a destination so as to minimize their own travel cost or travel time.

Rate: 6

With 2 roads,
\[\text{Cost}_a = \text{Cost}_b = 83\]

With 3 roads,
\[\text{Cost}_a = \text{Cost}_b = \text{Cost}_c = 92\]
Braess Paradoxes: definition

Pareto-inefficient equilibria can exhibit unexpected behavior.

**Definition: **Braess Paradox.

There is a Braess Paradox if there exists two systems $\text{ini}$ and $\text{aug}$ such that

$$\text{ini} < \text{aug \ and \ } \alpha^{(nc)}(\text{ini}) > \alpha^{(nc)}(\text{aug}).$$

i.e. adding resources to the system may reduce the performances of **ALL** players simultaneously.
From the New York Times, Dec 25, 1990, Page 38, What if They Closed 42d Street and Nobody Noticed?, By GINA KOLATA:

"ON Earth Day this year, New York City’s Transportation Commissioner decided to close 42d Street, which as every New Yorker knows is always congested. "Many predicted it would be doomsday," said the Commissioner, Lucius J. Riccio. "You didn’t need to be a rocket scientist or have a sophisticated computer queuing model to see that this could have been a major problem." But to everyone’s surprise, Earth Day generated no historic traffic jam. Traffic flow actually improved when 42d Street was closed."
Braess Paradoxes: applications
Non-cooperative scheduling with 1-port hypothesis

Hypothesis: the master can only send to 1 slave at a time.

Example

maître: $W = 2.55$
3 machines: $(B_i, W_i) = (4.12, 0.41), (4.61, 1.31), (3.23, 4.76)$
2 applications: $b^1 = 1, w^1 = 2, b^2 = 2, w^2 = 1$

Equilibrium (ini): $a^1 = 0.173, a^2 = 0.0365$
Equilibrium ($W_2 = 5.4$): $a^1 = 0.127, a^2 = 0.0168$
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Evolutionary games

- User strategies change with time as they adapt to the state
- Different possible dynamics:
  - Replicator dynamics:
    \[
    \dot{y}_q^s = y_q^s \left( F_q^s(y) - \frac{1}{\hat{d}_q} \sum_{i=1}^{S_q} y_q^i F_q^i(y) \right).
    \]
  - Brown von Neumann Nash Dynamics (BNN):
    \[
    \gamma_q^s = \max \left\{ F_q^s(y) - \frac{1}{\hat{d}_q} \sum_{i=1}^{S_q} y_q^i F_q^i(y), 0 \right\} \quad \text{(excess payoff)}
    \]
    \[
    \dot{y}_q^s = \hat{d}_q \gamma_q^s - y_q^s \sum_{j=1}^{S_q} \gamma_j^s.
    \]
    (increase proportionally to the excess payoff / decrease proportionally to the sum of excess payoffs)
Equilibria are called ESS (Evolutionary Stable Strategies) or
Subset of Nash equilibria
Stable by a deviation of a (small) fraction of users

Example of applications:
- Power choice in ALOHA systems
  - Users can choose to transmit at high or low power (each packet)
  - High power has better chances of not being jammed
  - Low power save battery consumption
- Associations in wireless systems
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3. Other yet interesting topics...
We only scratched the surface...

Many other frameworks of games:

- **Stackelberg equilibria:** strategic game between 2 players: a leader and a follower (used in pricing mechanisms of e-services). Over types used in pricing Bertrand competition, Cournot competition.

- **Stochastic games:** a type of dynamic games (i.e. evolving over time) where the transitions are stochastic - the next state is determined by a probability distribution depending on the current state and the chosen actions (Markov Decision Processes) (used to choose efficient scheduling strategies)
How to improve non-cooperative performance?

No universal solution, but several options:

**Correlated equilibria**:
- A correlator gives advice to each player
- (such that) the optimal strategy for each player is to follow the advice
- \( \text{Nash equilibria} \subset \text{Correlated equilibria} \)

Interestingly, studies have shown that in certain cases, the correlator does not need to have any information on the system.

**Pricing mechanisms**:
- An entity gives money (reward) to players
- Each player strives to maximize its profit

Problem well studied in TCP-like networks (based on Lagrangian optimization)
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3. **Other yet interesting topics...**
Axiomatic definition VS optimization problem

1. Pareto optimality
2. Symmetry
3. Invariance towards linear transformations

\[ \prod_{i} u_i \]

- Independant to irrelevant alternatives
  - Nash (NBS) / proportional fairness
  \[ \prod_{i} u_i \]

- Monotonicity
  - Raiffa-Kalai-Smorodinsky / max-min
  \[ \max \{ u_i | \forall j, u_i \leq u_j \} \]

- Inverse monotonicity
  - Thomson / Social welfare
  \[ \max \sum u_i \]
Fairness family [TAG06]

Introduced by Mo and Walrand

\[
\text{utility} \quad \max \sum_{n \in N} \frac{f_n \left( x_n \right)^{1-\alpha}}{1 - \alpha}
\]

player

Global Optimization
Proportional Fairness
Max–min Fairness
TCP Vegas
ATM (ABR)
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Fairness family: example
The COST network (Prop. fairness)
Fairness family: example
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Fairness family: example
CDMA wireless networks [AGT06]

Fair rate allocation: ex. AMR Codec (UMTS) allows 8 rates for voice (between 4.75 and 12.2 kbps) dynamically changed every 20ms.

Model uplink, downlink and macrodiversity

Challenge join allocation of throughput and power
Exemple $\alpha = 0$: global optimization
Exemple $\alpha = 2.5$
How to fairly allocate the bandwidth provided by a geostationary satellite among different network operators?

**System:** MF-TDMA (Multiple Frequency-Time Division Multiple Access), operators ask for a certain number of carriers of certain capacities.

**Constraints:**

- **Integrity constraints:** \( N \) types of carriers, of bandwidth \( B_1, B_2, \ldots, B_N \).
- **Inter-Sopt Compatibility Conditions (ISCC):**
  - (i) imposing the use of the same frequency plan on ALL spots of a same color
  - (ii) allowing to replace the demand of a client for a carrier \( j \) by a carrier \( t \) with \( t < j \).
Fairness family: example
MF-TDMA satellite networks [TAG03]

Without inter-spot constraint

With inter-spot constraint
Fairness family: example
MF-TDMA satellite networks [TAG03]
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3. Other yet interesting topics...
Two points (C and D) can be equally fair (symetrically identical).

\[ U_1 \]

\[ U_2 \]

\[ x^4 + y^4 = 1. \]
Two points (C and D) can be equally fair (symetrically identical).

A set of points cannot be differentiated by the $\alpha$-family.

$x^4 + y^4 = 1.015$
Other hot topics in game theory

- **Mechanism design**: how to design rules of a game so as to achieve a specific outcome, even though each player is selfish. Done by setting up a structure in which each player has incentive to behave as the designer intends. (Leonid Hurwicz, Eric Maskin et Roger Myerson, Nobel 2007)

- **Auctions**: resource allocation in P2P, frequency allocation in wireless.


- **Fair division or cake cutting problem**: how to divide resource such that all recipients believe that they have received their fair share (envy-free). (Steven Brams, Alan Taylor)
When multiple users have conflicting objectives cooperation is the way to go to achieve both fairness and efficiency.

But, individual users are prone to act selfishly, which can lead to catastrophic situations (Nash equilibria inefficiencies, Braess paradoxes...).

So, collaboration has to be induced (corelators, pricing mechanisms...) or enforced (penalties).
Example of enforced collaboration (set of rules enforced by the police)
While the purely non-cooperative approach would give...
Eitan Altman, Jérôme Galtier, and Corinne Touati.
Fair power and transmission rate control in wireless networks.
In IEEE/IFIP Third Annual Conference on Wireless On

Corinne Touati, Eitan Altman, and Jérôme Galtier.
Radio planning in multibeam geostationary satellite networks.
In AIAA International Communication Satellite Systems
Conference and Exhibit (ICSSC 2003), Yokohama, Japan, April 2003.

Corinne Touati, Eitan Altman, and Jérôme Galtier.
Generalized Nash bargaining solution for bandwidth allocation.

Corinne Touati, Hisao Kameda, and Atsushi Inoie.
Fairness in non-convex systems.
Slides available at:
http://www-id.imag.fr/~touati/Talks/GameTheory_07.pdf