

RIEMANNIAN GAME DYNAMICS

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ABSTRACT. We study a class of evolutionary game dynamics whose defining element is a Riemannian metric, which defines a state-dependent geometric structure on the set of population states. The replicator dynamics and the (Euclidean) projection dynamics are the archetypal examples of this class. Like these representative dynamics, all Riemannian game dynamics satisfy certain basic desiderata, including positive correlation and global convergence in potential games. Moreover, when the underlying Riemannian metric satisfies a Hessian integrability condition, the resulting dynamics preserve many further properties of the replicator and projection dynamics. We examine the close connections between Hessian game dynamics and reinforcement learning in normal form games, extending and elucidating a well-known link between the replicator dynamics and exponential reinforcement learning.

1. INTRODUCTION

Viewed abstractly, evolutionary game dynamics assign to every population game a dynamical system on the game’s set of population states. Under most such dynamics, the vector of motion at a given population state depends only on payoffs and behavior at that state, implying that changes in aggregate behavior are determined by current strategic conditions. Such dynamics can be understood as state-dependent rules for transforming current payoffs into feasible directions of motion.

In this paper, we introduce a family of evolutionary game dynamics that we call *Riemannian game dynamics*. The defining element of these dynamics is the notion of a *Riemannian metric*, a state-dependent inner product on the positive orthant used to evaluate displacements from each point, thereby inducing a geometric structure on the simplex.¹ This structure and the game’s payoff functions determine how the population’s behavior evolves over time.

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¹To be clear, a Riemannian metric is not a metric in the sense of measuring distances between points in a metric space, but it induces such a distance function in a canonical way (Lee, 1997).

Riemannian game dynamics can be constructed in two distinct but equivalent ways. In the first, the vector of motion z at a given state x balances a *gain* determined by the game's payoffs against a *cost of motion* determined by the underlying Riemannian metric. In the case of potential games, this gain is equal to the potential's rate of change when the state moves from x along z . More generally, the gain along z is a weighted sum of current payoffs, with the weight on each strategy's payoff given by the corresponding component of z . For its part, the cost of motion along displacement vector z is determined by the squared norm of z , as given by the underlying Riemannian metric at the current population state. Motion under Riemannian dynamics maximizes the difference between gains and costs, subject to an admissibility constraint. In the interior of the simplex, all tangent directions are admissible; at non-interior states however, admissibility depends on the behavior of the Riemannian metric near the boundary of the simplex, a point we return to below.

The second construction of our dynamics is purely geometric, and employs some fundamental notions from differential geometry. It requires two steps. To start, the game's payoff field is converted into a representative displacement vector field following a canonical, state-dependent rule determined by the underlying Riemannian metric. The resulting vector field is then projected to the closest admissible direction of motion, where the notion of closeness is also provided by the Riemannian metric.

The two archetypal examples of Riemannian game dynamics are the replicator dynamics (Taylor and Jonker, 1978) and the (Euclidean) projection dynamics (Nagurney and Zhang, 1997), both derived from fairly simple structures. First, the replicator dynamics are derived from the *Shahshahani metric* (Shahshahani, 1979), under which the cost of increasing a strategy's relative frequency in the population is inversely proportional to said frequency. Second, the projection dynamics are obtained by measuring the cost of change along a given displacement vector in the standard Euclidean fashion, independently of the population's current state. Other Riemannian metrics can be used in applications where different strategies have clear affinities, allowing the presence and performance of one strategy to positively influence the use of similar alternatives.

The metric's boundary behavior is the source of a fundamental dichotomy that is best explained by looking at our two prototypical examples above. Under the replicator dynamics: (i) the law of motion for every game is continuous; (ii) the set of utilized strategies remains constant along every solution trajectory; and (iii) the dynamics' rest points are the restricted equilibria of the game – the states at which all strategies in use earn the same payoff. In contrast, under the Euclidean projection dynamics: (i) the law of motion is typically discontinuous at the boundary of the simplex; (ii) the set of utilized strategies may change repeatedly along a single solution trajectory; and (iii) the dynamics' rest points are the Nash equilibria of the underlying game. Based on this behavior, we obtain a natural distinction between *continuous* and *discontinuous Riemannian dynamics*, each category sharing the boundary behavior of its prototype. In Section 4, we introduce a variety of examples of Riemannian dynamics from both classes.

A basic aim of our analysis is to demonstrate that many basic properties of the replicator and Euclidean projection dynamics extend to our substantially more

general setting. In Section 5, we show that Riemannian dynamics satisfy the basic desiderata for evolutionary game dynamics: they heed a payoff monotonicity condition known as *positive correlation*, and they converge to rest points in potential games. In the latter context, Riemannian game dynamics also provide a broad generalization of *Kimura’s maximum principle* (Kimura, 1958; Shahshahani, 1979). This principle states that when agents are matched to play a normal form common interest game, the replicator dynamics move in the direction of maximal increase in average payoffs, provided that lengths of displacement vectors are evaluated using the Shahshahani metric. Extending this principle, we observe that in any potential game, Riemannian dynamics track the direction of maximum increase in potential, assuming that displacements from each state are evaluated using the specified Riemannian metric.

Obtaining further results on stability, convergence, and global behavior requires additional structure on our dynamics – and hence on the underlying Riemannian metric. This structure is provided by an *integrability condition*. In prior work on game dynamics, such conditions have been imposed on the *vector fields* used to convert the strategies’ payoffs into vectors of choice probabilities.² By contrast, the integrability condition employed here is imposed on the *matrix field* that defines a Riemannian metric, requiring that it be expressible as the Hessian of a convex function. We call this function the *potential* of the metric, and we refer to the resulting dynamics as *Hessian game dynamics*.³ Both the replicator dynamics and the Euclidean projection dynamics are members of this class. As we explain in Section 6, Hessian dynamics are continuous when their potential function becomes infinitely steep at the boundary of the simplex. A version of this criterion separates Hessian dynamics into continuous and discontinuous types.

The key tool that we employ for the analysis of Hessian dynamics is the *Bregman divergence* of the metric’s potential (Bregman, 1967), an asymmetric measure of the “remoteness” of a given population state from any fixed target state.⁴ By using the Bregman divergence as a Lyapunov function, we prove global convergence to Nash equilibrium in strictly contractive games and local stability of evolutionarily stable states under Hessian game dynamics. We also show that certain distinctive properties of the replicator dynamics in normal form games extend to all continuous Hessian dynamics – in particular, the convergence of time averages of interior solutions to the set of Nash equilibria, and the existence of simple sufficient conditions for permanence. Finally, we show that strictly dominated strategies are eliminated under continuous Hessian dynamics, a conclusion which does not extend to the discontinuous regime.⁵

Related work. There are very close connections between the dynamics considered here and dynamics studied by Hofbauer and Sigmund (1990), Hopkins (1999), and

²See Hart and Mas-Colell (2001), Hofbauer and Sandholm (2007), and Sandholm (2010a).

³In the context of convex programming, gradient flows generated by Hessian Riemannian (HR) metrics of this sort have been explored at depth by Bolte and Teboulle (2003) and Alvarez et al. (2004). Laraki and Mertikopoulos (2015) also examine the long-term rationality properties of a class of second-order, *inertial* game dynamics derived from HR metrics.

⁴In the Shahshahani case, this boils down to the Kullback–Leibler divergence, which has seen wide use in the analysis of the replicator dynamics (Hofbauer and Sigmund, 1998; Weibull, 1995).

⁵See Sandholm et al. (2008) and Section 6.4.

Harper (2011). In order to have the machinery in place to make these connections clear, we postpone this discussion until Section 2.3.

There is a more surprising connection between Hessian dynamics and models of reinforcement learning in normal form games. Rustichini (1999), Hofbauer et al. (2009) and Mertikopoulos and Moustakas (2010) show that if players track the cumulative payoffs (or *scores*) of their strategies and choose mixed strategies at each instant by applying the logit choice rule to these scores, the evolution of mixed strategies is described by the replicator dynamics.⁶ Combining our analysis here with that in a companion paper (Mertikopoulos and Sandholm, 2016), we show that Hessian dynamics derived from a steep potential function also describe the evolution of mixed strategies under reinforcement learning. In addition to substantially generalizing existing results, our analysis provides an intuitive explanation for the tight links between the two processes. Section 7 describes these and other connections between Hessian dynamics and reinforcement learning in detail.

2. POPULATION GAMES AND EVOLUTIONARY DYNAMICS

Notation. Let $\mathcal{A} = \{\alpha_1, \dots, \alpha_n\}$ be a finite set. The real space spanned by \mathcal{A} will be denoted by $\mathbb{R}^{\mathcal{A}}$ and we will write $\delta_{\alpha\beta}$ for the Kronecker deltas on \mathcal{A} . We will also write $\mathcal{K} \equiv \mathbb{R}_+^{\mathcal{A}}$ for the nonnegative orthant of $\mathbb{R}^{\mathcal{A}}$, $\mathcal{K}^\circ \equiv \mathbb{R}_{++}^{\mathcal{A}}$ for its interior (the positive orthant), and $\mathbb{R}_0^{\mathcal{A}} = \{z \in \mathbb{R}^{\mathcal{A}} : \sum_{\alpha} z_{\alpha} = 0\}$ for the subspace of vectors whose components sum to zero. Finally, in a slight abuse of notation, we will write $\mathbb{R}^{\text{supp}(x)} = \{z \in \mathbb{R}^{\mathcal{A}} : z_{\alpha} = 0 \text{ whenever } x_{\alpha} = 0\}$ for the set of vectors in $\mathbb{R}^{\mathcal{A}}$ whose supports are contained in the support of $x \in \mathbb{R}^{\mathcal{A}}$.

2.1. Population games. Throughout this paper we focus on games played by a population of nonatomic agents. Our analysis extends to the multi-population setting without significant effort, but we focus on single-population games for simplicity and notational clarity.

During play, each agent chooses an *action* (or *pure strategy*) from a finite set \mathcal{A} , and their payoff is determined by their choice of action and by the proportions $x_{\alpha} \in [0, 1]$ of the population playing each action $\alpha \in \mathcal{A}$. Collectively, these proportions define a *population state* $x = (x_{\alpha})_{\alpha \in \mathcal{A}} \in \mathbb{R}^{\mathcal{A}}$, and we write $\mathcal{X} = \Delta(\mathcal{A}) = \{x \in \mathbb{R}_+^{\mathcal{A}} : \sum_{\alpha} x_{\alpha} = 1\}$ for the set of population states (or *state space*) of the game. The payoff to an agent playing $\alpha \in \mathcal{A}$ when the population state is $x \in \mathcal{X}$ is then given by an associated *payoff function* $v_{\alpha} : \mathcal{X} \rightarrow \mathbb{R}$, which we assume to be Lipschitz continuous. Putting all this together, a *population game* may be identified with a set of actions and their associated payoff functions, and will be denoted by $\mathcal{G} \equiv \mathcal{G}(\mathcal{A}, v)$.

A population state $x^* \in \mathcal{X}$ is a *Nash equilibrium* (NE) of a population game \mathcal{G} if

$$v_{\alpha}(x^*) \geq v_{\beta}(x^*) \quad \text{for all } \alpha \in \text{supp}(x^*) \text{ and for all } \beta \in \mathcal{A}. \quad (\text{NE})$$

If x^* satisfies (NE) and is *pure* (i.e. $x^* = e_{\alpha}$ for some $\alpha \in \mathcal{A}$), it is called a *pure Nash equilibrium* of \mathcal{G} ; if, in addition, (NE) holds as a strict inequality for all $\beta \notin \text{supp}(x^*)$, x^* is said to be a *strict equilibrium* of \mathcal{G} .

A *restriction* of a game \mathcal{G} is a population game $\mathcal{G}' \equiv \mathcal{G}'(\mathcal{A}', v')$ that is defined by a subset $\mathcal{A}' \subseteq \mathcal{A}$ of the original game's action set and by payoff functions v_{α} obtained by restricting the original payoff functions to the reduced state space $\mathcal{X}' = \Delta(\mathcal{A}')$ of \mathcal{G}' . If $x \in \mathcal{X}$ is a Nash equilibrium of some restriction of \mathcal{G} , it will be called a

⁶For related results, see also Börgers and Sarin (1997), Posch (1997), and Hopkins (2002).

restricted equilibrium; put differently, $x \in \mathcal{X}$ is a restricted equilibrium of \mathcal{G} if all strategies in its support earn equal payoffs.

Below we briefly discuss some classes of population games that have been studied extensively in the literature. The first is the focus of the early works on evolutionary game dynamics (Hofbauer and Sigmund, 1998; Weibull, 1995):

Example 2.1 (Matching in normal form games). The simplest example of a population game is obtained by uniformly matching a population of agents to play a two-player symmetric normal form game with payoff matrix $A = (A_{\alpha\beta})_{\alpha,\beta=1}^n$. Aggregating over all matches, the payoff to an α -strategist when the population is at state $x \in \mathcal{X}$ is $v_\alpha(x) = \sum_{\beta \in \mathcal{A}} A_{\alpha\beta} x_\beta$. Thus, uniform random matching generates payoff functions that are linear in the population state.

The next two classes of games have a wide variety of applications, and their Nash equilibria are the only stable limit points of a broad range of evolutionary dynamics (Sandholm, 2010b).

Example 2.2 (Potential games). A population game \mathcal{G} is called a *potential game* (Monderer and Shapley, 1996; Sandholm, 2001) if there exists a *potential function* f defined on a neighborhood of \mathcal{X} such that:⁷

$$\frac{\partial f}{\partial x_\alpha}(x) = v_\alpha(x) \quad \text{for all } \alpha \in \mathcal{A} \text{ and for all } x \in \mathcal{X}. \quad (2.1)$$

Potential games include congestion games, games generated by variable pricing schemes, and population games generated by random matching in common interest games ($A_{\alpha\beta} = A_{\beta\alpha}$).

Example 2.3 (Contractive games). A population game \mathcal{G} is called (*weakly*) *contractive*⁸ (Hofbauer and Sandholm, 2009) if

$$\sum_{\alpha} (v_\alpha(x') - v_\alpha(x))(x'_\alpha - x_\alpha) \leq 0 \quad \text{for all } x, x' \in \mathcal{X}. \quad (2.2)$$

If (2.2) binds only when $x = x'$, \mathcal{G} is called *strictly contractive*, whereas if (2.2) binds for all $x, x' \in \mathcal{X}$, \mathcal{G} is called *conservative*. Contractive games include wars of attrition, nonatomic congestion games with increasing cost functions, and games that admit a concave potential. Random matching in normal form games with an interior evolutionarily stable state generates strictly contractive population games (see Section 6.2), while random matching in symmetric zero-sum games ($A_{\alpha\beta} = -A_{\beta\alpha}$) generates conservative population games.

2.2. Evolutionary dynamics. The term *evolutionary dynamics* refers to rules that assign to each population game \mathcal{G} a dynamical system on its state space \mathcal{X} . This is usually done by mapping each game to a *law of motion*, i.e. a differential equation of the form

$$\dot{x} = V(x). \quad (\text{D})$$

In most cases, the *motion field* $V(x)$ of (D) is defined by introducing a mapping $(x, \pi) \mapsto \tilde{V}(x, \pi)$ from state/payoff pairs to vectors, and then specifying that $V(x) \equiv \tilde{V}(x, v(x))$. In what follows, we will focus exclusively on such dynamics.

⁷For most purposes, it suffices for f to be defined on \mathcal{X} and to ask that the directional derivative $f'(x; z)$ be equal to $\sum_{\alpha} v_\alpha(x) z_\alpha$ for all $z \in \mathbb{R}_0^A$ and all $x \in \mathcal{X}$ – see Sandholm (2009).

⁸Hofbauer and Sandholm (2009) use the name *stable games*, but Sandholm (2015) proselytizes for the terms employed here. In convex analysis, condition (2.2) is called *monotonicity*.

To ensure that solutions to (D) remain in \mathcal{X} for all $t \geq 0$, $V(x)$ should not point outward from \mathcal{X} ; formally, $V(x)$ should lie in the *tangent cone* of \mathcal{X} at x , defined here as

$$\text{TC}_{\mathcal{X}}(x) = \{z \in \mathbb{R}_0^A : z_\alpha \geq 0 \text{ whenever } x_\alpha = 0\}. \quad (2.3)$$

Under many evolutionary dynamics (including the replicator dynamics and other imitative dynamics), the support of $x(t)$ remains invariant under (D), so the interior of each face of \mathcal{X} remains invariant under (D). When this is the case, $V(x)$ actually lies in the *tangent space* to \mathcal{X} at x , defined as

$$\text{T}_{\mathcal{X}}(x) = \{z \in \mathbb{R}_0^A : z_\alpha = 0 \text{ whenever } x_\alpha = 0\} \subseteq \text{TC}_{\mathcal{X}}(x). \quad (2.4)$$

Clearly, for every interior state $x \in \mathcal{X}^\circ$, we have $\text{T}_{\mathcal{X}}(x) = \text{TC}_{\mathcal{X}}(x) = \mathbb{R}_0^A$.

A basic monotonicity criterion linking (D) with the underlying game requires *positive correlation* between the strategies' payoffs and growth rates. Formally, this means that

$$\sum_{\alpha} v_{\alpha}(x)V_{\alpha}(x) \geq 0 \quad \text{for all } x \in \mathcal{X}, \quad (\text{PC})$$

with equality only if $V(x) = 0$. If (D) satisfies (PC), every Nash equilibrium of \mathcal{G} is a rest point of (D). For a detailed discussion, see Sandholm (2010b).

We provide two prototypical examples of evolutionary dynamics below:

Example 2.4 (The replicator dynamics). The quintessential evolutionary game dynamics are the *replicator dynamics* of Taylor and Jonker (1978), described by the system

$$\dot{x}_{\alpha} = x_{\alpha} \left[v_{\alpha}(x) - \sum_{\beta} x_{\beta} v_{\beta}(x) \right]. \quad (\text{RD})$$

Under (RD), the percentage growth rate of a strategy in use is equal to the difference between its payoff and the population's average payoff; also, strategies that are initially unused remain so for all time.

Example 2.5 (The Euclidean projection dynamics). The other fundamental example we consider is the *Euclidean projection dynamics* of Nagurney and Zhang (1997) (see also Friedman, 1991, and Lahkar and Sandholm, 2008). These are defined as

$$\dot{x} = \arg \min_{z \in \text{TC}_{\mathcal{X}}(x)} \|v(x) - z\|_2^2, \quad (\text{PD})$$

where $\|z\|_2 = (\sum_{\alpha} z_{\alpha}^2)^{1/2}$ denotes the ordinary Euclidean norm on \mathbb{R}^A . Geometrically, the dynamics (PD) are defined by taking the Euclidean projection of the payoff field $v(x)$ onto the tangent cone $\text{TC}_{\mathcal{X}}(x)$. Since $\text{TC}_{\mathcal{X}}(x) = \mathbb{R}_0^A$ on the interior \mathcal{X}° of the simplex, we obtain the simple formula

$$\dot{x}_{\alpha} = v_{\alpha}(x) - \frac{1}{|\mathcal{A}|} \sum_{\beta \in \mathcal{A}} v_{\beta}(x), \quad (2.5)$$

valid for all interior $x \in \mathcal{X}^\circ$. According to (2.5), the absolute growth rate of a strategy equals the difference between its payoff and the unweighted average of all strategies' payoffs. For an explicit formula on the boundary of \mathcal{X} , see Example 4.2.

2.3. Antecedents. The class of dynamics studied here is a substantial generalization of both the replicator dynamics and the projection dynamics. We now describe works from an assortment of fields that are antecedents of our approach.

The replicator equation (RD) for common interest games is a basic model from population genetics (see Schuster and Sigmund, 1983). The *fundamental theorem of natural selection*, attributed to Fisher (1930), states that natural selection among genes increases overall population fitness. Kimura (1958) introduced a corresponding maximum principle, showing that population fitness increases at a maximum rate under (RD) provided that one imposes a certain nonlinear constraint on the set of feasible changes in population frequencies (see Remark 3.1 in Section 3.3). Later, Shahshahani (1979) and Akin (1979) put Kimura’s maximum principle on a firm mathematical footing using tools from differential geometry – specifically, by introducing a suitable Riemannian metric (see Section 3.1). The derivation of the replicator dynamics in the latter papers provides a basic instance of the geometric construction of Riemannian dynamics developed in Section 3.6, while our construction based on balancing gains and costs can be viewed as a variation on Kimura’s analysis (cf. Remark 3.1).

Hofbauer and Sigmund (1990) model natural selection in populations of animals whose traits are represented by elements of a continuous set. They assume that all members of the population share the same trait x , except for an infinitesimal group of mutants whose traits differ infinitesimally from x . The evolution of the preponderant trait x follows a gradient-like process, moving in the direction that agrees with the play of the most successful local mutants. To obtain variations on this process, Hofbauer and Sigmund (1990) use a Riemannian metric to define the size and shape of the neighborhood of local mutants. When the trait space is \mathcal{X} and the fitness of mutant y takes the linear form $\sum_{\alpha} y_{\alpha} v_{\alpha}(x)$, they showed that the evolution of x on the interior of \mathcal{X} is given by

$$\dot{x}_{\alpha} = \sum_{\beta \in \mathcal{A}} \left[g_{\alpha\beta}^{-1}(x) - \frac{\sum_{\gamma} g_{\alpha\gamma}^{-1}(x) \sum_{\gamma} g_{\gamma\beta}^{-1}(x)}{\sum_{\gamma, \kappa} g_{\gamma\kappa}^{-1}(x)} \right] v_{\beta}(x), \quad (2.6)$$

where $g(x)$ is a field of symmetric positive definite matrices that defines the Riemannian metric in question (see Section 3.1). Hofbauer and Sigmund (1990) then observed that under the Shahshahani metric, the system (2.6) boils down to the replicator dynamics (RD). As we shall see, (2.6) describes the dynamics studied in this paper at all states $x \in \mathcal{X}$ in what we call the *minimal-rank* case (cf. Section 3.4).

In the course of analyzing perturbed best response dynamics (Fudenberg and Levine, 1998) and variants of fictitious play (Brown, 1951), Hopkins (1999) introduced a class of game dynamics that are defined on the interior of \mathcal{X} as

$$\dot{x}_{\alpha} = \sum_{\beta \in \mathcal{A}} M_{\alpha\beta}(x) v_{\beta}(x). \quad (2.7)$$

Here $M(x)$ is a smoothly-varying field of symmetric matrices that are positive definite on $\mathbb{R}_0^{\mathcal{A}}$ and map constant vectors to 0. Hopkins (1999) showed that the linearization of these dynamics agrees with that of perturbed best response dynamics up to a positive affine transformation. As a result, the local stability of rest points of (2.7) agrees with that of the corresponding rest points of perturbed best response

dynamics with sufficiently small noise levels. As we show in [Appendix A.1](#), the dynamics (2.6) satisfy Hopkins' conditions, and, conversely, all dynamics satisfying Hopkins' conditions can be expressed in the form (2.6). Thus, on the interior of \mathcal{X} , the dynamics of Hopkins (1999) are equivalent to the dynamics studied here ([Proposition A.3](#)).

More recently, Harper (2011) used ideas from information geometry to define generalizations of the replicator dynamics, and employed concepts from Riemannian geometry to state and prove certain properties of the induced dynamics. Ignoring boundary issues, these dynamics are an important special case of ours – specifically, the class of separable dynamics that we introduce in [Example 4.4](#).

Finally, we note here that there is a surprising and deep connection between the Hessian subclass of Riemannian game dynamics and a model of reinforcement learning recently examined by Mertikopoulos and Sandholm (2016). We explore this relation in detail in [Section 7](#).

3. RIEMANNIAN GAME DYNAMICS

3.1. Riemannian metrics. To define the class of dynamics under study, we must first introduce some basic notions from Riemannian geometry.⁹ The fundamental distance-measuring concept is that of a *Riemannian metric*, a position-dependent variant of the ordinary (Euclidean) scalar product between vectors. To define it, recall first that a *scalar product* on $W = \mathbb{R}^A$ is a bilinear pairing $\langle \cdot, \cdot \rangle: W \times W \rightarrow \mathbb{R}$ which satisfies the following for all $w, w' \in W$:

- (1) *Symmetry*: $\langle w, w' \rangle = \langle w', w \rangle$.
- (2) *Positive-definiteness*: $\langle w, w \rangle \geq 0$, with equality if and only if $w = 0$.

The *norm* of a vector $w \in W$ is then defined as

$$\|w\| = \langle w, w \rangle^{1/2}. \quad (3.1)$$

To express the above in components, let $\{e_\alpha\}_{\alpha=1}^n$ denote the canonical basis of W . Then, if we write $w = \sum_\alpha w_\alpha e_\alpha$ and $w' = \sum_\beta w'_\beta e_\beta$, the bilinearity of $\langle \cdot, \cdot \rangle$ gives

$$\langle w, w' \rangle = \sum_\alpha \sum_\beta w_\alpha g_{\alpha\beta} w'_\beta = w^\top g w' \quad (3.2)$$

for some symmetric positive-definite matrix $g = (g_{\alpha\beta})_{\alpha, \beta \in \mathcal{A}}$. This matrix is called the *metric tensor* of $\langle \cdot, \cdot \rangle$, and its components are given by

$$g_{\alpha\beta} = \langle e_\alpha, e_\beta \rangle. \quad (3.3)$$

A scalar product is uniquely defined by its metric tensor and vice versa, so the two notions will be used interchangeably throughout.

With this in mind, a *Riemannian metric* on an open set U of W is defined as a C^1 -smooth assignment of scalar products $\langle \cdot, \cdot \rangle_x$ to each $x \in U$ – or, equivalently, as a C^1 field $g(x)$ of symmetric positive definite matrices on U . In other words, a Riemannian metric on U prescribes a way of measuring lengths of displacement vectors at each point $x \in U$. We provide two archetypal examples below:

Example 3.1. The *Euclidean metric* on $U = \mathbb{R}^A$ is defined by choosing $g(x)$ to be the identity matrix for all $x \in U$:

$$g(x) = I = \text{diag}(1, \dots, 1) \quad \text{for all } x \in U. \quad (3.4)$$

⁹For a comprehensive introduction to the topic, see the masterful account of Lee (1997, 2003).

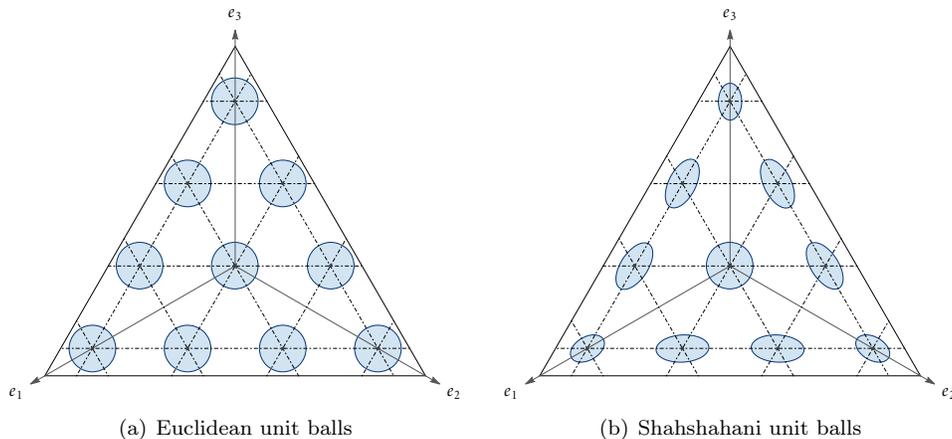


Figure 1. Unit balls on the 3-simplex under the Euclidean and Shahshahani metrics. For each base point x shown, the shaded regions comprise all tangent vectors w based at x that satisfy $\|w\|_x \leq 1$. When x_α is small, Shahshahani balls appear as ellipsoids that are “squashed” along the direction of the x_α -axis.

This definition yields the standard expressions $\langle w, w' \rangle_x = w^\top w'$ and $\|w\|_x = \sqrt{w^\top w}$, both independent of x .

Example 3.2. For a less trivial example, let $U = \mathcal{K}^\circ$ be the open positive orthant. Then the *Shahshahani metric* on U is defined as

$$g(x) = \text{diag}(1/x_1, \dots, 1/x_n) \quad \text{for all } x \in U. \quad (3.5)$$

Eq. (3.5) yields the Shahshahani inner product $\langle w, w' \rangle_x = \sum_\alpha w_\alpha w'_\alpha / x_\alpha$, which, in contrast to the Euclidean product, is x -dependent. For instance, since $\|e_\alpha\|_x = x_\alpha^{-1/2}$, the set of vectors at x with Shahshahani norm 1 is squeezed toward the x_α axis as x_α becomes small (see Fig. 1).

In our game-theoretic setting, a Riemannian metric can be interpreted as measuring the difficulty of motion away from any given population state. Specifically, when $z \in \text{TC}_\mathcal{X}(x)$ is a feasible vector of motion from population state $x \in \mathcal{X}$, we will refer to the quantity

$$C(z; x) = \frac{1}{2} \|z\|_x^2, \quad (3.6)$$

as the *cost of motion* along z from x . Defining costs of motion in terms of the squared norm makes them convex in z , with costs increasing quadratically as one increases the speed of motion in a fixed direction.

3.2. Duality: payoffs as covectors. We now discuss a fundamental duality between displacement vectors on \mathcal{X} (corresponding to feasible directions of motion) and linear functionals that act upon them. Specifically, a linear functional $\omega: W \rightarrow \mathbb{R}$ acting on vectors $w \in W$ is called a *covector*, and the space W^* of such functionals is called the *dual space* of W . We write $\langle \omega | w \rangle$ for the action of a covector $\omega \in W^*$ on a vector $w \in W$; to emphasize this pairing, the elements of W and W^* are also referred to as *primal* and *dual vectors* respectively. Also, using the standard

basis of W to write everything in matrix notation, vectors and covectors can be distinguished by writing primal vectors $w \in W$ as column vectors and dual vectors $\omega \in W^*$ as row vectors. The action $\langle \omega | w \rangle$ of ω on w is then given by the matrix product $\omega w = \sum_{\alpha} \omega_{\alpha} w_{\alpha}$.

In our game-theoretic context, the payoff “vector” $(v_{\alpha}(x))_{\alpha \in \mathcal{A}}$ acts as a linear functional on displacement vectors, and thus should be regarded as a *covector*, a point we illustrate in the examples to follow. Thus, when using matrix notation, we will represent the payoffs $v(x)$ as a *row vector*.

Example 3.3. The definition (NE) of a Nash equilibrium can be written as

$$\sum_{\alpha} v_{\alpha}(x^*)(x_{\alpha} - x_{\alpha}^*) \leq 0 \quad \text{for all } x \in \mathcal{X}, \quad (3.7)$$

or, in vector-covector notation,

$$\langle v(x^*) | x - x^* \rangle \leq 0 \quad \text{for all } x \in \mathcal{X}. \quad (3.8)$$

Since the displacement vector $z = x - x^*$ is tangent to \mathcal{X} , (3.8) only restricts the action of payoff covectors on the set of vectors $\mathbb{R}_0^{\mathcal{A}} = \{z \in \mathbb{R}^{\mathcal{A}} : \sum_{\alpha} z_{\alpha} = 0\}$ tangent to \mathcal{X}° . From a game-theoretic standpoint, this reflects the fact that Nash equilibria are not affected if payoffs to all strategies are shifted by the same amount.

Example 3.4. The defining property (2.1) of potential games can be expressed as

$$\langle df(x) | z \rangle = \langle v(x) | z \rangle \quad \text{for all } z \in \mathbb{R}^{\mathcal{A}} \text{ and all } x \in \mathcal{X}. \quad (3.9)$$

On the left-hand side, $df(x)$ denotes the *differential* of f at x , a linear functional that acts on tangent vectors $z \in \mathbb{R}^{\mathcal{A}}$ to yield the *directional derivative* $f'(x; z)$. Thus, (3.9) can be expressed as an equality between covectors, viz. $v(x) = df(x)$.

Example 3.5. In defining contractive games, the duality between payoffs and displacements in \mathcal{X} can be seen by rewriting (2.2) as

$$\langle v(x') | x' - x \rangle \leq \langle v(x) | x' - x \rangle \quad \text{for all } x, x' \in \mathcal{X}. \quad (3.10)$$

Example 3.6. Finally, the positive correlation condition (PC) can be written as:

$$\langle v(x) | V(x) \rangle \geq 0 \quad \text{for all } x \in \mathcal{X}, \quad (3.11)$$

with equality only when $V(x) = 0$.

In light of these examples, we will refer to the quantity

$$G(z; x) = \langle v(x) | z \rangle \quad (3.12)$$

as the *gain* of moving along z from population state $x \in \mathcal{X}$. In light of (3.9), the quantity $G(z; x)$ extends the notion of “the rate of increase in potential” to games that do not admit a potential function.¹⁰ In particular, the gain captures the alignment between the direction of motion z and the payoffs at state x ; it is also homogeneous in z , and so grows linearly as one increases the speed of motion in a fixed direction.

¹⁰The logic here is similar to an original motivation for the definition of contractive games, which extends the idea of a game with a concave potential function to games that do not admit a potential function – see Hofbauer and Sandholm (2009).

3.3. Riemannian game dynamics: the interior case. With these preliminaries in hand, we can give a partial definition of the game dynamics under study. To do so, fix a population game $\mathcal{G} \equiv \mathcal{G}(\mathcal{A}, v)$ and a Riemannian metric g defined on the positive orthant \mathcal{K}° . Then, on the set $\text{int}(\mathcal{X})$ of interior population states, the Riemannian game dynamics induced by g are defined as

$$\dot{x} = \arg \max_{z \in \mathbb{R}_0^A} [G(z; x) - C(z; x)] = \arg \max_{z \in \mathbb{R}_0^A} \left[\langle v(x) | z \rangle - \frac{1}{2} \|z\|_x^2 \right] \quad (3.13)$$

The vector z that solves the maximization problem in (3.13) balances the gain $G(z; x)$ of moving along z from x against the cost of motion $C(z; x)$ under the metric g – subject of course to the constraint that the population mass remain constant.

To obtain an explicit expression for (3.13), consider the associated Lagrangian

$$\Lambda(z, \mu; x) = \langle v(x) | z \rangle - \frac{1}{2} \|z\|_x^2 - \mu \sum_{\alpha \in \mathcal{A}} z_\alpha, \quad (3.14)$$

where the last term captures the feasibility constraint $\sum_{\alpha} z_\alpha = 0$. Then, writing $\mathbf{1} = (1, \dots, 1)$ for the row vector of ones in the standard basis of \mathbb{R}^A , a simple differentiation yields the first-order condition

$$v(x) = z^\top g(x) + \mu \mathbf{1}, \quad (3.15)$$

or, after rearranging,

$$z = g^{-1}(x) [v(x) - \mu \mathbf{1}]^\top, \quad (3.16)$$

where $g^{-1}(x)$ denotes the inverse of the matrix $g(x)$. Using the constraint $\sum_{\alpha} z_\alpha = 0$ to solve for μ and substituting in (3.13), some easy algebra leads to the explicit expression

$$\dot{x} = g^{-1}(x) \left[v(x) - \frac{\mathbf{1} g^{-1}(x) v(x)^\top}{\mathbf{1} g^{-1}(x) \mathbf{1}^\top} \mathbf{1} \right]^\top. \quad (3.17)$$

Thus, if we set

$$v^\sharp(x) = (v(x) g^{-1}(x))^\top \quad \text{and} \quad n(x) = (\mathbf{1} g^{-1}(x))^\top, \quad (3.18)$$

the dynamics (3.17) can be rewritten more concisely as

$$\dot{x} = v^\sharp(x) - \frac{\sum_{\beta} v_{\beta}^\sharp(x)}{\sum_{\beta} n_{\beta}(x)} n(x) = v^\sharp(x) - \frac{\langle v^\sharp(x), n(x) \rangle_x}{\|n(x)\|_x^2} n(x). \quad (3.19)$$

We now present the two archetypal examples of this class of dynamics:

Example 3.7. If $g(x) = I$ is the Euclidean metric, then $g^{-1}(x) = I$ as well. We thus get $v^\sharp(x) = v(x)^\top$ and $n(x) = \mathbf{1}^\top$, so (3.19) becomes

$$\dot{x}_\alpha = v_\alpha(x) - \frac{1}{|\mathcal{A}|} \sum_{\beta \in \mathcal{A}} v_\beta(x) \quad \text{for all } x \in \mathcal{X}^\circ. \quad (3.20)$$

Therefore, when costs of motion (3.6) are defined using Euclidean lengths, the components of the displacement vector \dot{x} equal those of the payoff covector $v(x)$ up to a constant that ensures that $\dot{x} \in \mathbb{R}_0^A$. The resulting system describes the (Euclidean) projection dynamics of Nagurney and Zhang (1997) on \mathcal{X}° (Example 2.5).

Example 3.8. If $g(x) = \text{diag}(1/x_1, \dots, 1/x_n)$ is the Shahshahani metric of [Example 3.2](#), then $g^{-1}(x) = \text{diag}(x_1, \dots, x_n)$. In this case, $v_\alpha^\sharp(x) = x_\alpha v_\alpha(x)$ and $n_\alpha(x) = x_\alpha$, so [\(3.19\)](#) becomes

$$\dot{x}_\alpha = x_\alpha \left[v_\alpha(x) - \sum_{\beta \in \mathcal{A}} x_\beta v_\beta(x) \right]. \quad (3.21)$$

When costs are defined using the Shahshahani metric, changes in the use of rare strategies are more costly than changes in the use of common ones. As a consequence, the initial term of \dot{x}_α is proportional to both the payoff $v_\alpha(x)$ of strategy α and to the mass x_α of agents playing strategy α . The second term ensures that $\dot{x} \in \mathbb{R}_0^{\mathcal{A}}$, but here the normalization for strategy α is itself proportional to x_α .

The system [\(3.21\)](#) describes the replicator dynamics of Taylor and Jonker (1978) ([Example 2.4](#)). Although we assumed that x is interior in [\(3.13\)](#), the expression [\(3.21\)](#) also describes the replicator dynamics on the boundary of \mathcal{X} for reasons we explain below.

Remark 3.1. The derivation in [Example 3.8](#) is closely related to Kimura’s (1958) derivation of the replicator dynamics in *common interest games*, i.e. games in which $v(x) = (Ax)^\top$ for some symmetric matrix A . Such games admit the potential function $f(x) = \frac{1}{2}x^\top Ax$, which is one-half of the population’s average payoff. Using somewhat different language, Kimura (1958) proposed the population dynamics

$$\dot{x} = \arg \max \{ \langle df(x)|z \rangle : z \in \text{TC}_{\mathcal{X}}(x) \text{ and } \|z\|_x^2 = \sigma_v^2(x) \}, \quad (3.22)$$

where $\|\cdot\|_x^2$ is the Shahshahani norm and $\sigma_v^2(x) = \sum_\alpha x_\alpha (v_\alpha(x) - \sum_\beta x_\beta v_\beta(x))^2$ denotes the variance in the population’s payoffs at state x . It is easy to verify that [\(3.22\)](#) boils down to the replicator dynamics for the potential game $v(x) = (Ax)^\top$.

3.4. Extensions to the boundary. We now turn to an important dichotomy that arises when extending the definition of the dynamics [\(3.13\)](#) to the boundary of \mathcal{X} . To begin, recall from [\(2.4\)](#) that the *tangent space* $\text{T}_{\mathcal{K}}(x)$ to the nonnegative orthant \mathcal{K} at x is the linear subspace

$$\text{T}_{\mathcal{K}}(x) = \{ z \in \mathbb{R}^{\mathcal{A}} : z_\alpha = 0 \text{ whenever } x_\alpha = 0 \} = \mathbb{R}^{\text{supp}(x)}. \quad (3.23)$$

We then say that a Riemannian metric g defined on \mathcal{K}° is *extendable* to \mathcal{K} if the map $x \mapsto g^{-1}(x)$ on \mathcal{K}° admits a (necessarily unique) C^1 -smooth extension to \mathcal{K} which we denote by g^\sharp (so $g^\sharp(x) \equiv g^{-1}(x)$ for all $x \in \mathcal{K}^\circ$), and which satisfies

$$\text{T}_{\mathcal{K}}(x) \subseteq \text{im } g^\sharp(x) \text{ for all } x \in \mathcal{K}. \quad (3.24)$$

In the above, $\text{im } g^\sharp(x)$ is the image (column space) of $g^\sharp(x)$; we henceforth call this set the *domain* of g at x and denote it by $\text{dom } g(x)$. [Proposition B.1](#) in [Appendix B](#) shows that if g is extendable in the sense of [\(3.24\)](#), then the field of scalar products associated with g also admits a unique continuous extension from \mathcal{K}° to \mathcal{K} , with $\langle \cdot, \cdot \rangle_x$ defined on $\text{dom } g(x)$.

In what follows, we focus on two basic forms of extendability. First, if $\text{dom } g(x) = \mathbb{R}^{\mathcal{A}}$ for all $x \in \mathcal{K}$, we say that g is *full-rank extendable*; instead, if $\text{dom } g(x) = \text{T}_{\mathcal{K}}(x)$ for all $x \in \mathcal{K}$, we say that g is *minimal-rank extendable*. We henceforth use the term “extendable” to refer to these two cases only.

Example 3.9. The Euclidean metric has $g^\sharp(x) = g^{-1}(x) = I$ for all $x \in \mathcal{K}$, so it is full-rank extendable by default.

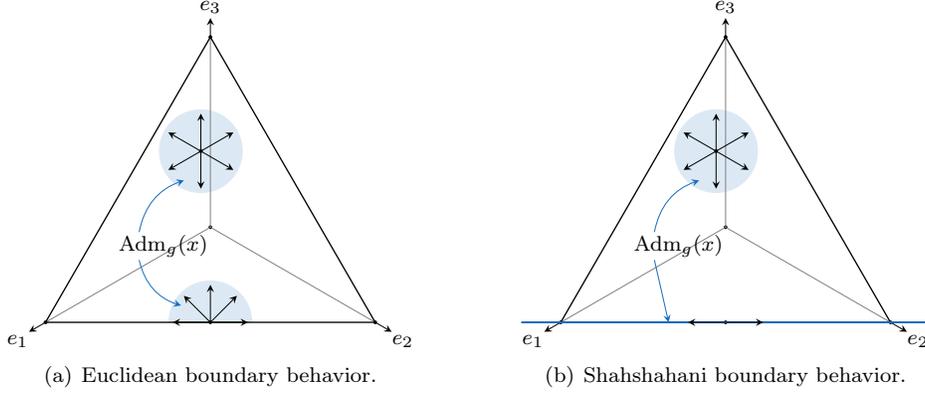


Figure 2. Admissible sets under the Euclidean and Shahshahani metrics. For $x \in \mathcal{X}^\circ$, we have $\text{Adm}_g(x) = \text{TC}_{\mathcal{X}}(x)$. For $x \in \text{bd}(\mathcal{X})$, we still have $\text{Adm}_g(x) = \text{TC}_{\mathcal{X}}(x)$ in the Euclidean case, but the Shahshahani metric can only be extended to the tangent space $\text{Adm}_g(x) = \text{T}_{\mathcal{X}}(x)$.

Example 3.10. The Shahshahani metric (3.5) has $g^\sharp(x) = g^{-1}(x) = \text{diag}(x_1, \dots, x_n)$, so $\text{dom } g(x) = \text{T}_{\mathcal{K}}(x) = \mathbb{R}^{\text{supp}(x)}$ for all $x \in \mathcal{K}$. Thus, the Shahshahani metric is minimal-rank extendable, and the induced scalar product on $\text{dom } g(x) = \mathbb{R}^{\text{supp}(x)}$ is

$$\langle w, w' \rangle_x = \sum_{\alpha \in \text{supp}(x)} w_\alpha w'_\alpha / x_\alpha \quad \text{for all } w, w' \in \text{T}_{\mathcal{K}}(x). \quad (3.25)$$

Remark 3.2. Intuitively, minimal-rank extendable metrics partition \mathcal{K} into the relative interiors of each of its faces (including \mathcal{K} itself). We will see that under the dynamics generated by such metrics, the relative interior of each face of \mathcal{X} is an invariant set.

To extend the definition of the dynamics to the boundary $\text{bd}(\mathcal{X})$ of \mathcal{X} , we introduce the cone of g -admissible vectors

$$\text{Adm}_g(x) = \text{TC}_{\mathcal{X}}(x) \cap \text{dom } g(x), \quad (3.26)$$

This cone, which comprises all tangent vectors $z \in \text{TC}_{\mathcal{X}}(x)$ that also lie in $\text{dom } g(x)$, specifies the possible directions of motion of our dynamics at a given state $x \in \mathcal{X}$. When $x \in \mathcal{X}^\circ$ is interior, we have $\text{TC}_{\mathcal{X}}(x) = \text{T}_{\mathcal{X}}(x) = \mathbb{R}_0^{\mathcal{A}}$ and $\text{dom } g(x) = \mathbb{R}^{\mathcal{A}}$; thus, the g -admissible set is the hyperplane

$$\text{Adm}_g(x) = \mathbb{R}_0^{\mathcal{A}} \cap \mathbb{R}^{\mathcal{A}} = \mathbb{R}_0^{\mathcal{A}}, \quad (3.27)$$

as anticipated in equation (3.13). Further instances of g -admissible cones are depicted in Fig. 2.

The restriction to $\text{dom } g(x)$ is needed because the norm $\|z\|_x^2$ is only defined for $z \in \text{dom } g(x)$. The only case in which $\text{dom } g(x)$ is not all of $\mathbb{R}^{\mathcal{A}}$ occurs when $x \in \text{bd}(\mathcal{X})$ and g is minimal-rank extendable, in which case $\text{dom } g(x) = \text{T}_{\mathcal{X}}(x) = \mathbb{R}^{\text{supp}(x)}$ (cf. Example 3.10).

3.5. Riemannian game dynamics. We are at last in a position to define Riemannian game dynamics throughout \mathcal{X} . Let $\mathcal{G} \equiv \mathcal{G}(\mathcal{A}, v)$ be a population game with payoff functions $v_\alpha : \mathcal{X} \rightarrow \mathbb{R}$, and let g be an extendable Riemannian metric on the positive

orthant \mathcal{K}° of \mathbb{R}^A . Then, building on (3.13), the *Riemannian game dynamics* induced by g are

$$\dot{x} = \arg \max_{z \in \text{Adm}_g(x)} \left[\langle v(x)|z \rangle - \frac{1}{2} \|z\|_x^2 \right] \quad (\text{RmD})$$

In Section 3.3, we showed that (RmD) can be expressed at interior states as

$$\dot{x} = v^\sharp(x) - \frac{\sum_\beta v_\beta^\sharp(x)}{\sum_\beta n_\beta(x)} n(x), \quad (3.28a)$$

where $v^\sharp(x) = g^\sharp(x)v(x)^\top$ and $n(x) = g^\sharp(x)\mathbb{1}^\top$. After a slight rearrangement, we can also express the dynamics as a linear function of the payoff covector $v(x)$:

$$\dot{x}_\alpha = \sum_{\beta \in \mathcal{A}} \left[g_{\alpha\beta}^\sharp(x) - \frac{n_\alpha(x)n_\beta(x)}{\sum_\gamma n_\gamma(x)} \right] v_\beta(x). \quad (3.28b)$$

Equation (A.2b) in Appendix A provides a concise third expression for the dynamics on \mathcal{X}° in terms of a pseudoinverse matrix.

If g is minimal-rank extendable, Proposition B.2 in Appendix B shows that (3.28) holds for all $x \in \mathcal{X}$, provided that one uses $g^\sharp(x)$ in the definition (3.18) of $v^\sharp(x)$ and $n(x)$. Proposition B.2 also shows that, in this case, one need only take the sums in the formulas (3.28) over the strategies in the support of x .

If instead g is full-rank extendable, extending (3.28) to boundary states requires solving a convex program whose inequality constraints may be active. For this reason, coordinate formulas for (RmD) may depend on the support of x – and, indeed, (RmD) may fail to be continuous at the boundary of \mathcal{X} (see Example 4.2 below). With this in mind, it will be convenient to call the dynamics generated by minimal-rank extendable metrics *continuous Riemannian dynamics*, and those generated by full-rank extendable metrics *discontinuous Riemannian dynamics*.

3.6. A geometric derivation. In (RmD), the dynamics' vector of motion from x is defined to maximize the difference between the gain $G(z; x) = \langle v(x)|z \rangle$ and the cost of motion $C(z; x) = \frac{1}{2} \|z\|_x^2$ over the set of admissible vectors $z \in \text{Adm}_g(x)$. We now show how these dynamics can be derived using a purely geometric approach, generalizing Shahshahani's (1979) derivation of the replicator dynamics in common interest games, and Nagurney and Zhang's (1997) definition of the Euclidean projection dynamics. In what follows, we rely on some basic ideas from Riemannian geometry; for a comprehensive treatment, we refer again to Lee (2003).

The basic aim of our derivation is to find a vector field $x \mapsto V(x) \in \text{TC}(x)$ that agrees to the greatest possible extent with the payoff covector field $x \mapsto v(x)$, where this "agreement" is defined in terms of the Riemannian metric g . The derivation requires two steps: *i*) using a canonical transformation to convert the covector field into a vector field; and *ii*) projecting this field onto the cone of admissible vectors of motion.

For the first step, fix a Riemannian metric g on \mathcal{K}° that is extendable to \mathcal{K} as discussed in Section 3.4. The *primal equivalent* of a covector $\omega \in W^*$ at $x \in \mathcal{K}$ is the (necessarily unique) vector $\omega^\sharp \in \text{dom } g(x)$ such that

$$\langle \omega|w \rangle = \langle \omega^\sharp, w \rangle_x \quad \text{for all } w \in \text{dom } g(x), \quad (3.29)$$

or, in matrix notation,

$$\omega w = (\omega^\sharp)^\top g(x)w. \quad (3.30)$$

It is easy to verify that

$$\omega^\sharp = (\omega g^\sharp(x))^\top, \quad (3.31)$$

in agreement with our definition of $v^\sharp(x) = (v(x)g^\sharp(x))^\top$ from (3.18).

For the second step, we transform each vector $v^\sharp(x)$ into a g -admissible vector by projecting it onto $\text{Adm}_g(x)$. Specifically, for all $x \in \mathcal{X}$ and $w \in \text{dom } g(x)$, the *projection* of w at x is defined as

$$\Pi_x(w) = \arg \min_{z \in \text{Adm}_g(x)} \|w - z\|_x. \quad (3.32)$$

The induced Riemannian dynamics are then defined as

$$\dot{x} = \Pi_x(v^\sharp(x)). \quad (3.33)$$

When $x \in \mathcal{X}^\circ$ is interior, we have $\text{Adm}_g(x) = \mathbb{R}_0^A$ by default, so $\Pi_x(w)$ is simply the orthogonal projection of $w \in \text{dom } g(x) = \mathbb{R}^A$ onto \mathbb{R}_0^A with respect to g . Accordingly, $\Pi_x(w)$ can be computed by finding a normal vector to \mathbb{R}_0^A and subtracting this vector's contribution to w (as in the first step of the Gram-Schmidt orthonormalization process). To carry this out, observe that $\langle \mathbf{1} | z \rangle = \sum_\alpha z_\alpha = 0$ for all $z \in \mathbb{R}_0^A$, so the vector $n(x) = (\mathbf{1}g^\sharp(x))^\top$ defined in equation (3.18) satisfies

$$\langle n(x), z \rangle_x = n(x)^\top g(x)z = \mathbf{1}g^\sharp(x)g(x)z = \langle \mathbf{1} | z \rangle = 0 \quad \text{for all } z \in \mathbb{R}_0^A. \quad (3.34)$$

This shows that $n(x)$ is a *normal vector* to \mathbb{R}^A at x with respect to $g(x)$. Thus, for all $x \in \mathcal{X}^\circ$, we can express the right-hand side of (3.33) as

$$\Pi_x(v^\sharp(x)) = v^\sharp(x) - \text{proj}_{n(x)} v^\sharp(x) = v^\sharp(x) - \frac{\langle n(x), v^\sharp(x) \rangle_x}{\|n(x)\|_x^2} n(x), \quad (3.35)$$

in agreement with (3.19).

More generally, for any state $x \in \mathcal{X}$ we have

$$\begin{aligned} \Pi_x(v^\sharp(x)) &= \arg \min_{z \in \text{Adm}_g(x)} \|v^\sharp(x) - z\|_x \\ &= \arg \min_{z \in \text{Adm}_g(x)} \left[\|v^\sharp(x)\|_x^2 + \|z\|_x^2 - 2\langle v^\sharp(x), z \rangle_x \right] \\ &= \arg \max_{z \in \text{Adm}_g(x)} \left[\langle v^\sharp(x), z \rangle_x - \frac{1}{2}\|z\|_x^2 - \frac{1}{2}\|v^\sharp(x)\|_x^2 \right] \\ &= \arg \max_{z \in \text{Adm}_g(x)} \left[\langle v(x) | z \rangle - \frac{1}{2}\|z\|_x^2 \right]. \end{aligned} \quad (3.36)$$

The dynamics (3.33) and (RmD) are therefore identical. We will take advantage of this geometric representation (RmD) freely in what follows.

Remark 3.3. In addition to building on Kimura's and Shahshahani's derivations of the replicator dynamics, the dual representations of Riemannian game dynamics have a close analogue in a class of game dynamics called *target projection dynamics* (Friesz et al., 1994; Sandholm, 2005). These dynamics are defined on \mathcal{X} as

$$\dot{x} = \arg \min_{x' \in \mathcal{X}} \|v(x) - x'\|_2^2 - x, \quad (3.37)$$

Using a version of (3.36), Tsakas and Voorneveld (2009) showed that (3.37) can also be expressed as

$$\dot{x} = \arg \max_{x' \in \mathcal{X}} \left[\langle v(x) | x' \rangle - \frac{1}{2}\|x' - x\|_2^2 \right] - x. \quad (3.38)$$

4. EXAMPLES

We now present a variety of examples of Riemannian game dynamics. We start by verifying that the two prototypical dynamics from Section 2 are indeed induced by Riemannian metrics. To do so, we extend Examples 3.7 and 3.8 to allow for boundary states.

Example 4.1 (The replicator dynamics). Let g be the Shahshahani metric, so $g_{\alpha\beta}^{\sharp}(x) = \delta_{\alpha\beta}x_{\beta}$, $n_{\alpha}(x) = x_{\alpha}$, and $v_{\alpha}^{\sharp}(x) = x_{\alpha}v_{\alpha}(x)$. Then, (3.28) yields the (continuous) Riemannian dynamics

$$\dot{x}_{\alpha} = x_{\alpha} \left[v_{\alpha}(x) - \sum_{\beta} x_{\beta} v_{\beta}(x) \right], \quad (\text{RD})$$

which are the replicator dynamics of Taylor and Jonker (1978). The dynamics' continuity is reflected in the fact that the formula (RD) is valid throughout \mathcal{X} .

Example 4.2 (The Euclidean projection dynamics). Let g be the Euclidean metric. Starting from formulation (3.33), Lahkar and Sandholm (2008) derived the following representation of the associated (discontinuous) Riemannian dynamics:

$$\dot{x}_{\alpha} = \begin{cases} v_{\alpha}(x) - |\mathcal{A}(x)|^{-1} \sum_{\beta \in \mathcal{A}(x)} v_{\beta}(x) & \text{if } \alpha \in \mathcal{A}(x), \\ 0 & \text{otherwise,} \end{cases} \quad (4.1)$$

where $\mathcal{A}(x)$ is a subset of \mathcal{A} that maximizes the average $|\mathcal{A}'|^{-1} \sum_{\beta \in \mathcal{A}'} v_{\beta}(x)$ over all subsets $\mathcal{A}' \subset \mathcal{A}$ that contain $\text{supp}(x)$. These are the projection dynamics (PD) of Nagurney and Zhang (1997). The discontinuity of (PD) is reflected in the appearance of $\text{supp}(x)$ in (4.1) via the definition of $\mathcal{A}(x)$.

Remark 4.1. The dynamics (RD) and (PD) highlight an important qualitative difference between Shahshahani and Euclidean projections, which is representative of continuous and discontinuous Riemannian dynamics respectively. The replicator dynamics (RD) comprise a Lipschitz continuous dynamical system on \mathcal{X} which preserves the face structure of \mathcal{X} , in that the relative interior of each face of \mathcal{X} remains invariant. By contrast, the projection dynamics (PD) may fail to be continuous at the boundary of \mathcal{X} . Thus, the relevant notion of a solution to (PD) is that of a *Carathéodory solution*, which allows for kinks at a measure zero set of times. As a result, solutions of (PD) may leave and re-enter the relative interior of any face of \mathcal{X} in perpetuity.

The following example is an immediate generalization of the two dynamics discussed above:

Example 4.3 (The p -replicator dynamics). Given $p \geq 0$, we define the p -replicator dynamics as the Riemannian dynamics induced by the metric

$$g(x) = \text{diag}(1/x_1^p, \dots, 1/x_n^p), \quad (4.2)$$

There are three values of p worth highlighting:

- (1) For $p = 0$, we obtain the projection dynamics (PD).
- (2) For $p = 1$, we obtain the replicator dynamics (RD).
- (3) For $p = 2$, substitution into (3.28) yields the *log-barrier dynamics*

$$\dot{x}_{\alpha} = x_{\alpha}^2 \left(v_{\alpha}(x) - \frac{\sum_{\beta} x_{\beta}^2 v_{\beta}(x)}{\sum_{\beta} x_{\beta}^2} \right), \quad (\text{LD})$$

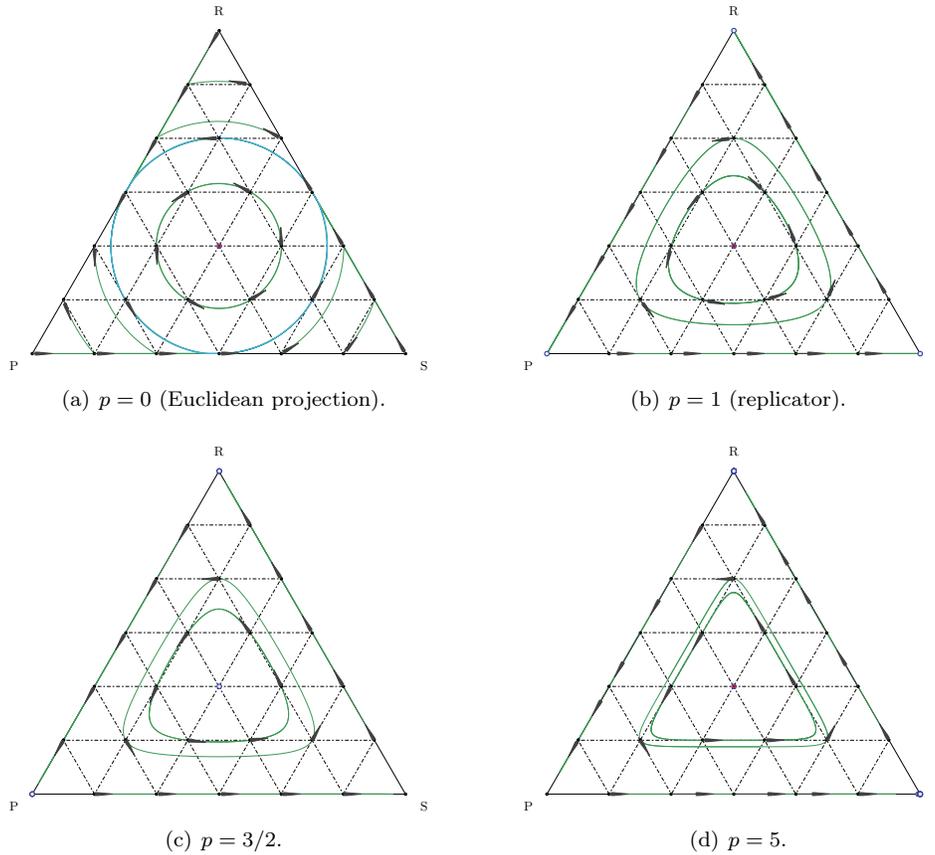


Figure 3. Phase portraits of the p -replicator dynamics in standard Rock-Paper-Scissors. As $p \in [0, \infty)$ increases, the shape of the closed orbits changes from circular to triangular. When $p = 0$, solutions enter and leave the boundary of the simplex, but forward solutions exist and are unique. For $p \geq 1$, forward and backward solutions exist, are unique, and their support is constant.

a system first examined by Bayer and Lagarias (1989) in the context of convex programming.

Under p -replicator dynamics, the value of p parametrizes the costs of changes in the use of less common strategies. This is illustrated in Fig. 3, which presents a collection of p -replicator phase portraits in standard Rock-Paper-Scissors:

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}. \quad (4.3)$$

When $p = 0$, displacement costs are independent of the current state; thus the circular form of the payoffs (4.3) generates circular closed orbits, subject to feasibility constraints (Fig. 3(a)). As p increases, the costs of motion for uncommon strategies become more important relative to the game's payoffs (Figs. 3(b) and 3(c)).

As a direct consequence, the closed orbits of the dynamics are “flattened” near each face of the simplex, and are ultimately reshaped into a nearly triangular form (Fig. 3(d)).¹¹

Fig. 3 also illustrates a basic dichotomy between continuous and discontinuous Riemannian dynamics. In the discontinuous case ($p = 0$), there is a unique forward solution from every initial condition in \mathcal{X} . However, solutions may enter and leave the boundary of \mathcal{X} , and solutions from different initial conditions can merge in finite time. In the smooth regime ($p \geq 1$), solutions exist and are unique in forward and backward time, and the support of the state remains fixed along each solution trajectory. Existence and uniqueness of solutions is treated formally in Section 5.1.

Example 4.4 (Separable metrics and their dynamics). A Riemannian metric g on \mathcal{K}° is called *separable* if its metric tensor is of the form

$$g(x) = \text{diag}(1/\phi(x_1), \dots, 1/\phi(x_n)), \quad (4.4)$$

where $\phi: [0, \infty) \rightarrow [0, \infty)$ is a continuous *weighting function* that is strictly positive on $(0, \infty)$. We then get

$$g^\sharp(x) = \text{diag}(\phi(x_1), \dots, \phi(x_n)), \quad (4.5)$$

so g is minimal-rank extendable if $\lim_{z \rightarrow 0^+} \phi(z) = 0$ and full-rank extendable otherwise.

When (3.28) applies, the dynamics induced by g take the form

$$\dot{x}_\alpha = \phi(x_\alpha) \left(v_\alpha(x) - \frac{\sum_{\beta \in \mathcal{A}} \phi(x_\beta) v_\beta(x)}{\sum_{\beta \in \mathcal{A}} \phi(x_\beta)} \right). \quad (4.6)$$

Ignoring the dynamics’ boundary behavior, (4.6) was studied by Harper (2011) under the name *escort replicator dynamics*, and was further examined by Mertikopoulos and Sandholm (2016) and Bravo and Mertikopoulos (2016) in the context of game-theoretic learning (see Section 7). It is clear that the construction above generalizes immediately to allow different weighting functions for different strategies.

Moving beyond the separable case, Riemannian dynamics can capture the effects of intrinsic relationships among the game’s strategies.

Example 4.5 (Nested replicator dynamics). Let $\mathcal{A}_1, \dots, \mathcal{A}_m$ be a partition of \mathcal{A} into m groups of intrinsically similar strategies, let $[\alpha]$ denote the group containing strategy α , let $x_{[\alpha]} = \sum_{\beta \in [\alpha]} x_\beta$, and, let $d_0, d_1 > 0$. Then, consider the Riemannian metric with metric tensor

$$g_{\alpha\beta}(x) = \begin{cases} d_0 \frac{\delta_{\alpha\beta}}{x_\alpha} + d_1 \frac{1}{x_{[\alpha]}} & \text{if } \beta \in [\alpha], \\ 0 & \text{otherwise.} \end{cases} \quad (4.7)$$

Under this metric, the cost of motion parallel to a face of the simplex is given by

$$C(e_\beta - e_\alpha; x) = \begin{cases} \frac{1}{2} d_0 \left(\frac{1}{x_\alpha} + \frac{1}{x_\beta} \right) & \text{if } \beta \in [\alpha], \\ \frac{1}{2} d_0 \left(\frac{1}{x_\alpha} + \frac{1}{x_\beta} \right) + \frac{1}{2} d_1 \left(\frac{1}{x_{[\alpha]}} + \frac{1}{x_{[\beta]}} \right) & \text{otherwise.} \end{cases} \quad (4.8)$$

Thus switches between strategies in different groups are more costly than switches between strategies in the same group.

¹¹That all of these dynamics feature closed orbits is not coincidental – see Proposition 6.5.

A straightforward calculation shows that

$$g_{\alpha\beta}^{\#}(x) = \begin{cases} \frac{1}{d_0}x_{\alpha}\delta_{\alpha\beta} - \frac{1}{d_0} \frac{d_1}{d_0+d_1} \frac{x_{\alpha}x_{\beta}}{x_{[\alpha]}} & \text{if } \beta \in [\alpha], \\ 0 & \text{otherwise.} \end{cases} \quad (4.9)$$

It is evident from (4.9) that the metric g is minimal-rank extendable. Thus by defining $\lambda_1 = \frac{1}{d_0} \frac{d_1}{d_0+d_1}$ and $\lambda_2 = \frac{1}{d_0+d_1}$ and applying (3.28), we find that g generates the *nested replicator dynamics*:

$$\dot{x}_{\alpha} = x_{\alpha} \left[\lambda_1 \left(v_{\alpha}(x) - \frac{1}{x_{[\alpha]}} \sum_{\beta \in [\alpha]} x_{\beta} v_{\beta}(x) \right) + \lambda_2 \left(v_{\alpha}(x) - \sum_{\beta \in \mathcal{A}} x_{\beta} v_{\beta}(x) \right) \right] \quad (\text{NRD})$$

if $x_{\alpha} > 0$ and $\dot{x}_{\alpha} = 0$ otherwise.

The imitative dynamics (NRD) were introduced by Mertikopoulos and Sandholm (2017) to model settings in which agents assess strategies using two distinct procedures: at rate λ_1 , an agent only compares the payoff of his current strategy α to those of strategies in group $[\alpha]$; at rate λ_2 , he compares the payoff of his current strategy to those of all other strategies. More general specifications allow for multiple levels of nesting; see Mertikopoulos and Sandholm (2017) for details.

The following class of dynamics incorporates all of our previous examples. It is examined at depth in Sections 6 and 7.

Example 4.6 (Hessian Riemannian metrics and their dynamics). A generalization of the above class of examples can be obtained by considering Riemannian metrics that are defined as Hessians of convex functions.¹² To that end, let $h: \mathcal{K} \rightarrow \mathbb{R}$ be a continuous function on \mathcal{K} such that

- (i) h is C^3 -smooth on every positive suborthant of \mathcal{K} .
- (ii) $\text{Hess } h(x)$ is positive-definite for all $x \in \mathcal{K}^{\circ}$.

Then, h induces a natural Riemannian metric on \mathcal{K}° defined as

$$g = \text{Hess } h, \quad (4.10)$$

or, in components:

$$g_{\alpha\beta}(x) = \frac{\partial^2 h(x)}{\partial x_{\alpha} \partial x_{\beta}}. \quad (4.11)$$

When this is the case, we say that g is a *Hessian Riemannian* (HR) metric and we refer to h as the *potential* of g .

The metric (4.7) that generates the nested replicator dynamics (NRD) is an HR metric with potential

$$h(x) = \sum_{\alpha \in \mathcal{A}} x_{\alpha} (d_0 \log x_{\alpha} + d_1 \log x_{[\alpha]}). \quad (4.12)$$

Moreover, every separable metric of the form (4.4) is an HR metric with potential

$$h(x) = \sum_{\alpha \in \mathcal{A}} \theta(x_{\alpha}) \quad (4.13)$$

¹²For the origins of the idea in geometry, see Duistermaat (2001), and references therein; for applications to convex programming, see Bolte and Teboulle (2003) and Alvarez et al. (2004).

p	NAME	REGULARITY	POTENTIAL
0	projection	discontinuous	quadratic
(0, 1)	—	not Lipschitz	power law
1	replicator	smooth	Gibbs entropy
(1, 2)	—	smooth	Tsallis entropy
2	log-barrier	smooth	logarithmic
(2, ∞)	—	smooth	inverse power law

Table 1. Regularity of the p -replicator dynamics and behavior of the generating potential function h_p .

for some smooth function $\theta: [0, +\infty) \rightarrow \mathbb{R}$ with $1/\theta''(z) = \phi(z)$. In particular, for $p \neq 1, 2$, the p -replicator dynamics are generated by the potential

$$h_p(x) = \sum_{\alpha \in \mathcal{A}} \theta_p(x_\alpha) \quad \text{with} \quad \theta_p(z) = \frac{1}{(p-1)(p-2)} z^{2-p}, \quad (4.14)$$

and when $p \neq 1, 2$, the potentials are

$$h_1(x) = \sum_{\alpha \in \mathcal{A}} x_\alpha \log x_\alpha \quad \text{and} \quad h_2(x) = - \sum_{\alpha \in \mathcal{A}} \log x_\alpha, \quad (4.15)$$

respectively.¹³ The values $p = 0, 1, 2$ partition the class of p -replicator dynamics into six cases whose properties we summarize in Table 1.¹⁴

Definition (4.10) is an integrability condition on the matrix field g . As with vector fields on simply connected domains, integrability of matrix fields can be characterized by a symmetry condition on its derivatives, namely that¹⁵

$$\frac{\partial g_{\alpha\gamma}}{\partial x_\beta} = \frac{\partial g_{\beta\gamma}}{\partial x_\alpha} \quad \text{for all } \alpha, \beta, \gamma \in \mathcal{A}. \quad (4.16)$$

Conditions (4.10) and (4.16) differ fundamentally from integrability conditions appearing in previous work on game dynamics, which are imposed on the *vector fields* that define the dynamics.¹⁶

In Section 6, we show that this integrability property provides important theoretical tools for the analysis of the induced Riemannian dynamics

$$\dot{x} = \arg \max_{z \in \text{Adm}_g(x)} \left[\langle v(x) | z \rangle - \frac{1}{2} \|z\|_x^2 \right], \quad g = \text{Hess } h, \quad (\text{HD})$$

which we call *Hessian game dynamics*.

¹³It is possible to define the potential h_p for all values of p using a single formula. Let $\theta_p(z) = (z^{2-p} + p(p-2)z - (p-1)^2)/((p-1)(p-2))$ when $p \neq \{1, 2\}$, and define θ_1 and θ_2 by analytic continuation. Linear and constant terms do not affect the resulting metric, and the explicit formulas for θ_1 and θ_2 follow from the fact that $\lim_{a \rightarrow 0} (z^a - 1)/a = \log z$.

¹⁴When $p \geq 2$, the potential h_p becomes infinite on the boundary of \mathcal{K} , violating a standing assumption; we address this technicality in Section 6. Also, the (negative) *Tsallis entropy* (Tsallis, 1988) mentioned in Table 1 is defined as $S_q(x) = (q-1)^{-1} \sum_{\alpha} (x_\alpha^q - x_\alpha)$ for some $q \in (0, 1)$.

¹⁵This characterization follows from the integrability condition for ordinary vector fields (i.e. symmetry of the Jacobian matrix) and the symmetry of $g(x)$.

¹⁶See Hart and Mas-Colell (2001), Hofbauer and Sandholm (2009), and Sandholm (2010a, 2014).

5. GENERAL PROPERTIES

In this section, we present some of our general results for (RmD). In Section 5.1 we state a basic but technically challenging result on the existence and uniqueness of solutions. In Section 5.2 we show that the dynamics exhibit positive correlation with the game's payoffs, and we characterize the dynamics' rest points as either restricted equilibria or Nash equilibria. Finally, in Section 5.3 we study the global behavior of the dynamics in potential games.

5.1. Existence and uniqueness of solutions. To illustrate the possibilities for existence and uniqueness of solutions, it is useful to start with a simple example. Specifically, consider the p -replicator dynamics of Example 4.3 for a 2-strategy game with action set $\mathcal{A} = \{1, 2\}$ and payoff functions $v_1(x) = 1$, $v_2(x) = 0$.

When $p = 1$, we obtain the toy replicator equation

$$\dot{x}_1 = x_1(1 - x_1). \quad (5.1)$$

Solutions to this equation exist and are unique for all $t \in (-\infty, \infty)$, and the support of $x(t)$ is invariant. The pure states 0 and 1 are both rest points, and it is easy to check that the unique solution with initial condition $x_1(0) = a \in (0, 1)$ is $x_1(t) = a/[a + (1 - a)e^{-t}]$.

When $p = 0$, we obtain the Euclidean projection dynamics

$$\dot{x}_1 = \begin{cases} 1/2 & \text{if } x_1 < 1 \\ 0 & \text{if } x_1 = 1. \end{cases} \quad (5.2)$$

For every initial condition $x_1(0) \in [0, 1]$, this equation admits the unique forward solution $x_1(t) = x_1(0) + t/2$ for $t \in [0, 2(1 - x_1(0))]$ and $x_1(t) = 1$ thereafter. Evidently, the support of $x(t)$ is not invariant; also, backward solutions are not defined for all time, and solutions are not smooth in t when $x_1 = 1$ is reached.

Finally, when $p = 1/2$, we obtain the differential equation

$$\dot{x}_1 = \frac{\sqrt{x_1(1 - x_1)}}{\sqrt{x_1} + \sqrt{1 - x_1}}. \quad (5.3)$$

Although this equation admits forward (and backward) solutions from every initial condition, these are no longer unique. Starting at $x_1(0) = 0$, we have the stationary solution $x_1(t) = 0$ for $t \in [0, \infty)$; furthermore, one can verify by a direct – albeit tedious – calculation that there is another solution, namely $x_1(t) = \frac{1}{2} + \frac{t-2}{4}\sqrt{1+t-t^2/4}$ for $t \in [0, 4)$ and $x_1(t) = 1$ thereafter. Additional solutions may linger at $x_1 = 0$ before emulating the previous solution trajectory.

The differences in behavior in the three cases above can be traced back to the properties of the underlying Riemannian metrics. First, the replicator dynamics are generated by the Shahshahani metric, which is minimal-rank extendable to all of \mathcal{X} . In this case the induced dynamics (RmD) are Lipschitz continuous, so existence and uniqueness of solutions is guaranteed by the Picard–Lindelöf theorem (along with an argument to account for \mathcal{X} being closed). Moreover, the support of $x(t)$ is constant, and solutions exist in both forward and backward time (Sandholm, 2010b, Theorems 4.A.5 and 5.4.7).

On the other hand, the Euclidean projection dynamics (PD) are generated by a full-rank extendable metric. In such cases, the induced dynamics (RmD) are typically discontinuous, so the relevant solution notion is that of a *Carathéodory*

solution, an absolutely continuous trajectory that satisfies (RmD) for almost all $t \geq 0$. In the case of (PD), Lahkar and Sandholm (2008) showed that every initial condition admits a unique Carathéodory forward solution; however, different solution orbits can merge in finite time, as illustrated in the previous example and in Fig. 3(a).

The following proposition shows that this behavior of (RD) and (PD) is representative of the minimal-rank and full-rank extendable cases respectively:

Proposition 5.1. *Let g be an extendable Riemannian metric.*

- (i) *If g is minimal-rank extendable, (RmD) admits a unique global solution from every initial condition in \mathcal{X} ; moreover, each solution has constant support.*
- (ii) *If g is full-rank extendable, (RmD) admits a unique forward Carathéodory solution from every initial condition in \mathcal{X} .*

Proposition 5.1 justifies the terminology *continuous* and *discontinuous* that we introduced in Section 3.5 to refer to dynamics induced by minimal-rank and full-rank metrics. The nontrivial part of Proposition 5.1 is the proof of part (ii): despite an apparent similarity, this result is considerably harder than the corresponding result of Lahkar and Sandholm (2008) for (PD), so we relegate its proof to Appendix D. The main reason for this difficulty is that known uniqueness proofs for projected differential equations depend crucially on the Riemannian metric being constant throughout the dynamics' state space, an assumption that obviously fails here.

Of course, as can be seen from the continuous – but not *Lipschitz* continuous – system (5.3), (RmD) may fail to admit unique solutions from initial conditions at the boundary of \mathcal{X} if the underlying metric does not admit a *Lipschitz* continuous extension to the boundary of \mathcal{X} . To avoid the resulting complications, we do not treat such pathologies in the rest of our paper.

5.2. Basic properties. We now establish some basic relationships between (RmD) and the payoffs of the underlying game. We first show that (RmD) respects the positive correlation condition (PC):

Proposition 5.2. *The dynamics (RmD) satisfy (PC).*

Proof. Let $V(x) = \Pi_x(v^\sharp(x))$. We then claim that

$$\langle v(x)|V(x) \rangle = \langle v^\sharp(x), V(x) \rangle_x \geq \langle \Pi_x(v^\sharp(x)), V(x) \rangle_x = \|V(x)\|_x^2 \geq 0, \quad (5.4)$$

with equality if and only if $V(x) = 0$. The only step in (5.4) needing justification is the first inequality. We split the analysis into three cases. First, if $x \in \mathcal{X}^\circ$, the inequality binds because Π_x orthogonally projects \mathbb{R}^A onto $\mathbb{R}_0^A = \text{Adm}_g(x)$, which contains $V(x)$. Second, if $x \in \text{bd}(\mathcal{X})$ and g is minimal-rank extendable, then $v^\sharp(x) \in \mathbb{R}^{\text{supp}(x)}$, so the inequality binds because Π_x projects $\mathbb{R}^{\text{supp}(x)}$ orthogonally onto $\mathbb{R}_0^A \cap \mathbb{R}^{\text{supp}(x)} = \text{Adm}_g(x)$, which contains $V(x)$. Finally, if $x \in \text{bd}(\mathcal{X})$ and g is full-rank extendable, Π_x is the closest point projection of \mathbb{R}^A onto the tangent cone $\text{TC}_{\mathcal{X}}(x)$. Hence, Moreau's decomposition theorem (Hiriart-Urruty and Lemaréchal, 2001) implies that $v^\sharp(x) - \Pi_x(v^\sharp(x))$ lies in the normal cone

$$\text{NC}_{\mathcal{X}}(x) = \{w \in \mathbb{R}^A : \langle w, z \rangle_x \leq 0 \text{ for all } z \in \text{TC}_{\mathcal{X}}(x)\}. \quad (5.5)$$

Since $V(x) \in \text{Adm}_g(x) = \text{TC}_{\mathcal{X}}(x)$, the first inequality in (5.4) is immediate. \square

Proposition 5.2 is not particularly surprising: after all, the basic postulate behind (RmD) is that the dynamics' vector of motion is the closest feasible approximation to the game's payoff field, with the notion of closeness determined by the underlying Riemannian metric. As we show below, this alignment can be exploited further to characterize the dynamics' rest points.

Indeed, recall that one of the main attributes of the Euclidean projection dynamics (PD) is *Nash stationarity*:

$$x^* \in \mathcal{X} \text{ is a rest point if and only if it is a Nash equilibrium.} \quad (\text{NS})$$

This property does not hold under the replicator dynamics: for instance, every pure state of \mathcal{X} is stationary under (RD). In this case, (NS) is replaced by the notion of *restricted stationarity*:¹⁷

$$x^* \in \mathcal{X} \text{ is a rest point if and only if it is a restricted equilibrium.} \quad (\text{RS})$$

Our next result shows that this difference between the projection and the replicator dynamics is representative of the discontinuous and continuous cases, and highlights one advantage of the former over the latter:

Proposition 5.3.

- (i) *Continuous Riemannian dynamics satisfy (RS).*
- (ii) *Discontinuous Riemannian dynamics satisfy (NS).*

Proof. For (i), recall that the coordinate expression (3.28) for (RmD) always holds when g is minimal-rank extendable, and $\dot{x}_\alpha = 0$ whenever $x_\alpha = 0$. Therefore, it suffices to check that $x^* \in \mathcal{X}^\circ$ is a rest point if and only if all the components of $v(x^*)$ are equal. To that end, note that x^* is a rest point of (3.28) if and only if

$$v^\sharp(x^*) = \frac{\sum_\gamma v_\gamma^\sharp(x^*)}{\sum_\gamma n_\gamma(x^*)} n(x^*) \propto n(x^*). \quad (5.6)$$

In turn, this means that x^* is a rest point of (RmD) if and only if $v^\sharp(x^*) \propto n(x^*)$; our claim then follows from the fact that $g^\sharp(x^*)$ is invertible.

For (ii), assume that g is full-rank extendable and fix some $x^* \in \mathcal{X}$. By the variational characterization (3.8) of Nash equilibria, x^* is a Nash equilibrium if and only if

$$0 \leq \langle v(x^*) | x - x^* \rangle = \langle v^\sharp(x^*), x - x^* \rangle_{x^*} \quad \text{for all } x \in \mathcal{X}, \quad (5.7)$$

i.e. if and only if $v^\sharp(x^*)$ lies in the normal cone $\text{NC}_{\mathcal{X}}(x^*)$ of \mathcal{X} at x^* (cf. Eq. 5.5 above). Moreau's decomposition theorem then yields $v^\sharp(x^*) \in \text{NC}_{\mathcal{X}}(x^*)$ if and only if $\Pi_{x^*}(v^\sharp(x^*)) = 0$, so our assertion follows. \square

Remark 5.1. We note without proof that shifting all strategies' payoffs by the same amount has no effect on (RmD), and rescaling all strategies' payoffs by the same factor only changes the speed at which solution paths are traversed. In addition, on the face of \mathcal{X} spanned by a subset \mathcal{A}' of \mathcal{A} , continuous dynamics are invariant to changes in the payoffs of strategies outside of \mathcal{A}' .

¹⁷Recall here that x^* is a *restricted equilibrium* if all strategies in its support earn equal payoffs.

5.3. Global convergence in potential games. Recall that population game $\mathcal{G} \equiv \mathcal{G}(\mathcal{A}, v)$ is a potential game if $v_\alpha(x) = \partial_\alpha f(x)$ for some potential function $f: \mathcal{X} \rightarrow \mathbb{R}$ (cf. [Example 2.2](#)). It then follows from [Proposition 5.2](#) that f is a *strict global Lyapunov function* for [\(RmD\)](#), meaning that its value increases along [\(RmD\)](#) whenever the dynamics are not at rest.¹⁸ For continuous Riemannian dynamics, a standard Lyapunov argument then implies that all ω -limit points of [\(RmD\)](#) are rest points – and hence, by [Proposition 5.3](#), restricted equilibria of \mathcal{G} .

However, this argument does not extend to discontinuous dynamics and Nash equilibria, because it requires continuity of solutions with respect to initial conditions, a difficult requirement to establish in our case. To circumvent this obstacle, we establish a lower semi-continuous (l.s.c.) bound on the rate of change of the game’s potential function. This bound then allows us to apply [Proposition C.1](#) in [Appendix C](#), which shows that, for dynamics on a compact set, such a bound on the rate of change of a Lyapunov function guarantees global convergence.

Proposition 5.4. *Let \mathcal{G} be a potential game with potential function f . Then, f is a strict Lyapunov function for [\(RmD\)](#) and every ω -limit point of [\(RmD\)](#) is a rest point of [\(RmD\)](#). These are restricted equilibria if [\(RmD\)](#) is continuous, and Nash equilibria if [\(RmD\)](#) is discontinuous.*

Proof. Let $V(x) = \Pi_x(v^\sharp(x))$ and let $x(t)$ be a solution of [\(RmD\)](#). Then, [Proposition 5.2](#) yields

$$\frac{d}{dt}f(x(t)) = \langle df(x(t))|\dot{x}(t) \rangle = \langle v(x(t))|V(x(t)) \rangle \geq 0, \quad (5.8)$$

with equality if and only if $V(x(t)) = 0$. Hence, f is a strict global Lyapunov function for [\(RmD\)](#).

When [\(RmD\)](#) is (Lipschitz) continuous, a standard argument shows that every ω -limit point of [\(RmD\)](#) is a rest point thereof (see e.g. [Sandholm, 2010b](#), [Theorem 7.B.3](#)). The discontinuous case however requires a different treatment. To start, note that

$$\frac{d}{dt}f(x(t)) = \langle v(x(t))|V(x(t)) \rangle = \langle v^\sharp(x), \Pi_x(v^\sharp(x)) \rangle_x \geq \|V(x(t))\|_x^2 \geq 0, \quad (5.9)$$

where the first inequality follows from Moreau’s decomposition theorem. Both inequalities bind if and only if $V(x(t)) = 0$; since the speed function $x \mapsto \|V(x)\|_x$ is lower semi-continuous (cf. [Lemma C.2](#)), [Proposition C.1](#) shows that every ω -limit point of [\(RmD\)](#) is a rest point. \square

The classic analyses of [Kimura \(1958\)](#) and [Shahshahani \(1979\)](#) showed that in common interest games, average payoffs are increased at a maximal rate under the replicator dynamics, provided that “maximal” is defined with respect to the Shahshahani metric. We conclude this section by deriving an analogous principle for all Riemannian game dynamics. To state it, define the *gradient* of a smooth function $f: \mathcal{K}^\circ \rightarrow \mathbb{R}$ with respect to g by

$$\text{grad } f(x) = (df(x)g^{-1}(x))^\top, \quad (5.10)$$

that is, as the (necessarily unique) vector satisfying

$$\langle df(x)|z \rangle = \langle \text{grad } f(x), z \rangle_x \quad \text{for all } z \in \mathbb{R}^{\mathcal{A}}, x \in \mathcal{K}^\circ. \quad (5.11)$$

¹⁸Definitions concerning stability and convergence are collected in [Appendix C](#).

Geometrically, the vector $\text{grad} f(x)$ represents the direction of maximal increase of the function f at x with respect to the metric g .¹⁹ We then have:

Proposition 5.5. *Let \mathcal{G} be a potential game with potential function f and let g be an extendable Riemannian metric. Then, for all $x \in \mathcal{X}^\circ$, the vector field that defines (RmD) is the projection of $\text{grad} f$ onto $T_{\mathcal{X}}(x)$ with respect to g .*

Proof. Since $v(x) = df(x)$, we have $\Pi_x(v^\sharp(x)) = \Pi_x(\text{grad} f(x))$, as claimed. \square

Hence, at interior states, the dynamics (RmD) increase the value of potential at a maximal rate under the geometry defined by g , subject to feasibility. For discontinuous dynamics, this conclusion remains true even at boundary states. For continuous dynamics, the interior of each face of \mathcal{X} is invariant under (RmD), so this conclusion holds provided that feasibility is understood to incorporate this additional constraint.

6. HESSIAN GAME DYNAMICS

By virtue of the integrability property that defines them, potential games have desirable convergence properties under a wide range of evolutionary dynamics. By contrast, convergence results for other classes of games – for instance, contractive games and games with an evolutionarily stable state (ESS) – require additional structure, often taking the form of integrability properties built into the dynamics themselves.²⁰ In this section, we show that the integrability of Hessian Riemannian metrics allows us to generalize several properties of the replicator dynamics and the Euclidean projection dynamics to the class of Hessian game dynamics introduced in Section 4. Conversely, this analysis pinpoints the fundamental integrability property that underlies a variety of convergence results for the replicator and projection dynamics.

A key element of our analysis is the so-called *Bregman divergence*, which we introduce in Section 6.1. Section 6.2 establishes global convergence to equilibrium in contractive games and local stability of ESSs, while Section 6.3 demonstrates the convergence of time-averages of interior trajectories to Nash equilibrium and provides sufficient conditions for permanence. Finally, Section 6.4 establishes the elimination of strictly dominated strategies under continuous Hessian dynamics.

6.1. Bregman divergences. When used as a tool for establishing convergence, Lyapunov functions typically measure some sort of “distance” between the current state and a target state x^* . For Hessian dynamics, a natural point of departure is the potential function h of the metric $g = \text{Hess} h$ that defines them. However, since the (game-specific) target state x^* is independent of g , there is no reason that h itself should serve as a Lyapunov function. Instead, taking advantage of the convexity of h , we consider the difference between $h(x^*)$ and the best linear approximation of $h(x^*)$ from x .

Formally, the *Bregman divergence* of h (Bregman, 1967) is defined as

$$D_h(x^*, x) = h(x^*) - h(x) - h'(x; x^* - x), \quad x^*, x \in \mathcal{X}, \quad (6.1)$$

¹⁹Specifically, this means that $\text{grad} f(x) = \arg \max\{D_z f(x) : \|z\|_x = 1\}$; that this is so follows from the definition of $\text{grad} f(x)$ and the Cauchy-Schwarz inequality.

²⁰See Hofbauer and Sandholm (2007) and Sandholm (2010a).

where $h'(x; x^* - x)$ is the one-sided derivative of h at x along $x^* - x$, i.e.

$$h'(x; x^* - x) = \lim_{t \rightarrow 0^+} t^{-1} [h(x + t(x^* - x)) - h(x)]. \quad (6.2)$$

Since h is convex, we have

$$D_h(x^*, x) \geq 0, \quad \text{with equality if and only if } x^* = x. \quad (6.3)$$

On the other hand, D_h is not symmetric in x^* and x , so it is not a bona fide distance function on \mathcal{X} ; rather, $D_h(x^*, x)$ describes the remoteness of x from the base point x^* , hence the name ‘‘divergence’’.

Revisiting our two archetypal examples, the Euclidean metric is generated by the quadratic potential $h(x) = \frac{1}{2} \sum_{\alpha} x_{\alpha}^2$. Definition (6.1) then yields the *Euclidean divergence*

$$D_{\text{Eucl}}(x^*, x) = \frac{1}{2} \sum_{\alpha} (x_{\alpha} - x_{\alpha}^*)^2, \quad (6.4a)$$

which is (uncharacteristically) symmetric in x^* and x . Analogously, the Shahshahani metric is generated by the (negative) entropy $h(x) = \sum_{\alpha} x_{\alpha} \log x_{\alpha}$. A short calculation shows that the corresponding divergence function is the *Kullback–Leibler* (KL) divergence

$$D_{\text{KL}}(x^*, x) = \sum_{\alpha: x_{\alpha}^* > 0} x_{\alpha}^* \log(x_{\alpha}^*/x_{\alpha}), \quad (6.4b)$$

which has been used extensively in the analysis of the replicator dynamics (Hofbauer and Sigmund, 1998; Weibull, 1995).

The key qualitative difference between the Euclidean divergence (6.4a) and the KL divergence (6.4b) is that the former is finite for all $x, x^* \in \mathcal{X}$, whereas the latter blows up to $+\infty$ when $\text{supp}(x^*) \not\subseteq \text{supp}(x)$. The reason for this blow-up is that the entropy function $h(x) = \sum_{\alpha} x_{\alpha} \log x_{\alpha}$ becomes infinitely steep as any boundary point x of \mathcal{X} is approached from the interior of \mathcal{X} , i.e.

$$\sup_{\alpha \in \mathcal{A}} |\partial_{\alpha} h(x_n)| \rightarrow \infty \quad \text{for every interior sequence } x_n \text{ converging to } x. \quad (6.5)$$

When this is the case for all $x \in \text{bd}(\mathcal{X})$, we say that h is *steep* (Alvarez et al., 2004; Hofbauer and Sandholm, 2002). At the opposite end of the spectrum, if $dh(x)$ exists for all $x \in \mathcal{X}$, we say that h is *nonsteep*.

The link between the steepness of h and the finiteness of the associated Bregman divergence is provided by the following lemma:

Lemma 6.1. *Fix $x^* \in \mathcal{X}$ and let $\mathcal{D}(x^*)$ denote the union of the relative interiors of the faces of \mathcal{X} that contain x^* , i.e.*

$$\mathcal{D}(x^*) \equiv \{x \in \mathcal{X} : \text{supp}(x^*) \subseteq \text{supp}(x)\}. \quad (6.6)$$

If h is steep, we have $D_h(x^, x) < \infty$ for all $x \in \mathcal{D}(x^*)$; by contrast, if h is nonsteep, we have $D_h(x^*, x) < \infty$ for all $x \in \mathcal{X}$.*

Proof. If h is steep and $x \in \mathcal{D}(x^*)$, the smoothness of h on the face of \mathcal{X} spanned by $\text{supp}(x) \supseteq \text{supp}(x^*)$ implies that the directional derivative $h'(x; x^* - x)$ exists and is finite, so $D_h(x^*, x)$ is itself finite. If instead h is nonsteep, $h'(x; x^* - x)$ exists and is finite for all $x \in \mathcal{X}$, so again $D_h(x^*, x) < \infty$. \square

Beyond the positive-definiteness property (6.3), the attribute of the Bregman divergence that recommends it as a Lyapunov function for (HD) is that the level sets of $D_h(x^*, \cdot)$ are perpendicular to all rays emanating from x^* under $g = \text{Hess } h$. Formally, we have:

Lemma 6.2. *Let $g = \text{Hess } h$ be an extendable HR metric and let $x^* \in \mathcal{X}$. Then, for every smooth curve $x(t)$ with constant support containing that of x^* , we have:*

$$\frac{d}{dt} D_h(x^*, x(t)) = \langle \dot{x}(t), x(t) - x^* \rangle_{x(t)}. \quad (6.7)$$

In particular, if $D_h(x^, x(t))$ is constant, $\dot{x}(t)$ is perpendicular to $x(t) - x^*$. Finally, if h is nonsteep, the above conclusions hold for every smooth curve $x(t)$ on \mathcal{X} .*

Proof. The proof is a direct application of the chain rule:

$$\begin{aligned} \frac{d}{dt} D_h(x^*, x) &= - \sum_{\alpha} \left[\frac{\partial h}{\partial x_{\alpha}} \dot{x}_{\alpha} + \frac{\partial h}{\partial x_{\alpha}} \frac{d}{dt} (x_{\alpha}^* - x_{\alpha}) + \sum_{\beta} \frac{\partial^2 h}{\partial x_{\alpha} \partial x_{\beta}} (x_{\alpha}^* - x_{\alpha}) \dot{x}_{\beta} \right] \\ &= \sum_{\alpha, \beta} (x_{\alpha}^* - x_{\alpha}) g_{\alpha\beta}(x) \dot{x}_{\beta} = \langle \dot{x}, x - x^* \rangle_x, \end{aligned} \quad (6.8)$$

where all summations are taken over the (constant) support $\mathcal{A}' \equiv \text{supp}(x(t))$ of $x(t)$ and we used the fact that $\dot{x}_{\alpha} = 0$ for $\alpha \notin \mathcal{A}'$. Finally, in the nonsteep case, h is smooth throughout \mathcal{X} , so the above holds for every smooth curve $x(t)$. \square

Within the class of Hessian Riemannian metrics, steepness of h roughly corresponds to minimal-rank extendability of the metric $g = \text{Hess } h$, and nonsteepness to full-rank extendability. These analogies fail when the steepness of h does not adequately control the regularity of g near the boundary of \mathcal{X} , or when g is minimal-rank extendable but generates non-Lipschitz dynamics. Bearing this in mind, we use the term *continuous Hessian dynamics* for Riemannian dynamics generated by a minimal-rank extendable metric $g = \text{Hess } h$ with steep h , and the term *discontinuous Hessian dynamics* for Riemannian dynamics generated by a full-rank extendable metric $g = \text{Hess } h$ with nonsteep h . In what follows, we will tacitly assume that the dynamics (HD) are either continuous or discontinuous.

6.2. Contractive games and evolutionarily stable states. Recall that a population game $\mathcal{G} \equiv \mathcal{G}(\mathcal{A}, v)$ is called *contractive* if $\langle v(x') - v(x), x' - x \rangle \leq 0$ for all $x, x' \in \mathcal{X}$, *strictly contractive* if the inequality is strict whenever $x \neq x'$, and *conservative* if the inequality always binds (cf. Example 2.3). As is well known, the set of Nash equilibria of any contractive game is convex, and every strictly contractive game admits a unique Nash equilibrium (Hofbauer and Sandholm, 2009).

Combining the defining inequality of strictly contractive games with the variational characterization of Nash equilibria (3.8), it follows that the (necessarily unique) Nash equilibrium of a strictly contractive game satisfies the inequality

$$\langle v(x), x - x^* \rangle \leq 0 \quad \text{with equality only if } x = x^*. \quad (6.9)$$

Hofbauer and Sandholm (2009) call a state satisfying (6.9) a *globally evolutionarily stable state* (GESS). This is the global version of the seminal local solution concept of Maynard Smith and Price (1973): if (6.9) holds for all $x \neq x^*$ in a neighborhood of x^* , then x^* is called an *evolutionarily stable state* (ESS).²¹

It is well known that the GESS x^* of a strictly contractive game attracts all solutions of the replicator dynamics whose initial support contains that of x^* ; by comparison, x^* attracts *all* orbits of the Euclidean projection dynamics (PD). Theorem 6.3 extends these results to all Hessian dynamics (HD).

²¹This concise characterization of evolutionary stability is due to Hofbauer et al. (1979).

Theorem 6.3. *Let \mathcal{G} be the (necessarily unique) Nash equilibrium of a strictly contractive game \mathcal{G} . Then:*

- (i) *For all continuous Hessian dynamics, $D_h(x^*, \cdot)$ is a strict decreasing Lyapunov function on $\mathcal{D}(x^*)$, and x^* is asymptotically stable with basin $\mathcal{D}(x^*)$.*
- (ii) *For all discontinuous Hessian dynamics, $D_h(x^*, \cdot)$ is a strict decreasing global Lyapunov function, and x^* is globally asymptotically stable.*

Proof. We begin with the continuous case. By [Proposition 5.1\(i\)](#), every solution $x(t)$ of [\(HD\)](#) has constant support. Hence, if $x(0) \in \mathcal{D}(x^*)$, [Lemma 6.2](#) yields:

$$\begin{aligned} \frac{d}{dt} D_h(x^*, x) &= \langle \dot{x}, x - x^* \rangle_x = \langle \Pi_x(v^\sharp(x)), x - x^* \rangle_x \\ &= \langle v^\sharp(x), x - x^* \rangle_x = \langle v(x) | x - x^* \rangle, \end{aligned} \tag{6.10a}$$

$$\leq 0, \tag{6.10b}$$

where we used the definition of Π_x for minimal-rank extendable metrics (cf. [Section 3.6](#)) to obtain [\(6.10a\)](#) and the definition [\(6.9\)](#) of a GESS for [\(6.10b\)](#). Since equality in [\(6.10\)](#) holds if and only if $x = x^*$, we conclude that $D_h(x^*, x)$ is a strict Lyapunov function on $\mathcal{D}(x^*)$. If we can show in addition that $x(t)$ has no ω -limit points in $\mathcal{X} \setminus \mathcal{D}(x^*)$, then asymptotic stability with basin $\mathcal{D}(x^*)$ follows from standard arguments (see e.g. [Sandholm \(2010b\)](#), Theorem 7.B.3).

Assume therefore that $x(t)$ admits an ω -limit point $x^\omega \neq x^*$, so $x(t_n) \rightarrow x^\omega$ for some sequence of times $t_n \uparrow \infty$. Since $|\dot{x}_\alpha(t)|$ is bounded from above by $V_{\max} \equiv \sup_{x \in \mathcal{X}} \max_\beta |V_\beta(x)| < \infty$, there exists an open neighborhood U of x^ω and positive $a, \delta, n_0 > 0$ such that $x(t) \in U$ and $\langle v(x(t)) | x(t) - x^* \rangle \leq -a < 0$ for all $t \in [t_n, t_n + \delta]$ and all $n \geq n_0$. Hence, by [\(6.10\)](#), we get

$$D_h(x^*, x(t_n + \delta)) - D_h(x^*, x(0)) \leq \int_0^{t_n + \delta} \langle v(x(s)) | x(s) - x^* \rangle ds \leq -a(n - n_0)\delta. \tag{6.11}$$

We thus get $\liminf_{t \rightarrow \infty} D_h(x^*, x(t)) = -\infty$, a contradiction.

For the discontinuous case, note first that since $x^* - x \in \text{TC}_{\mathcal{X}}(x)$, Moreau's decomposition theorem implies that $\langle \Pi_x(v^\sharp(x)), x - x^* \rangle_x \leq \langle v^\sharp(x), x - x^* \rangle_x$. Thus, replacing the first equality in [\(6.10a\)](#) by an inequality, [\(6.10\)](#) shows that $D_h(x^*, \cdot)$ is a strict global Lyapunov function for [\(HD\)](#). Global asymptotic stability then follows from [Proposition C.1](#). \square

The only implication of \mathcal{G} being strictly contractive used in the previous proof is that its Nash equilibrium is a GESS. More generally, if a game admits an ESS x^* , applying the above arguments in a neighborhood of x^* defined by a level set of $D_h(x^*, \cdot)$ yields the following result:

Theorem 6.4. *Evolutionarily stable states are asymptotically stable under [\(HD\)](#).*

[Hofbauer and Sigmund \(1990\)](#), [Hopkins \(1999\)](#), and [Harper \(2011\)](#) all offer results on the local stability of *interior* evolutionarily stable states under Riemannian game dynamics. [Hofbauer and Sigmund \(1990\)](#) showed that under all Riemannian (not necessarily Hessian) dynamics ([RmD](#)), the function $L(x) = \langle x - x^*, x - x^* \rangle_{x^*}$ is a strict local Lyapunov function for interior ESSs x^* , implying that x^* is asymptotically stable. Likewise, [Hopkins \(1999\)](#) used linearization to establish local stability of *regular* interior ESSs ([Taylor and Jonker, 1978](#)) under [\(RmD\)](#). Finally, [Harper](#)

(2011) employed a version of the argument above to prove asymptotic stability of ESSs for separable Hessian dynamics of the form (4.4).

An important case of contractive games that do not admit an ESS is the class of conservative games, which include population games generated by matching in symmetric zero-sum games. Under the replicator dynamics, the KL divergence does not provide a strict Lyapunov function for conservative games, but rather a constant of motion. The following result extends this conclusion to all Hessian dynamics:

Proposition 6.5. *Let x^* be a Nash equilibrium of a conservative game \mathcal{G} . Then, $D_h(x^*, \cdot)$ is a constant of motion along any interior solution segment of (HD).*

Proof. Simply note that (6.10) binds if \mathcal{G} is conservative and $x \in \mathcal{X}^\circ$. \square

Remark 6.1. In the definition of (HD), we required that h be finite throughout \mathcal{X} . This requirement is unnecessary for the preceding results when x^* is interior; however, if x^* lies on the boundary of \mathcal{X} , the proofs of Theorems 6.3 and 6.4 do not go through because $D_h(x^*, \cdot)$ is no longer well-defined throughout $\mathcal{D}(x^*)$. Nevertheless, the results themselves remain true if $g = \text{Hess } h$ is separable, allowing us to handle the p -replicator dynamics for $p \geq 2$ (Example 4.3). To prove this, it suffices to replace the implicit summation in $h'(x; x^* - x)$ over all strategies with a sum extending over only the strategies that lie in the support of x^* .

6.3. Convergence of time-averaged trajectories and permanence. We now extend two classic results for the replicator dynamics in random matching games (Example 2.1) to Hessian dynamics. The results for these games take advantage of the linearity of payoffs $v_\alpha(x) = \sum_{\beta \in \mathcal{A}} A_{\alpha\beta} x_\beta$ in the population state.

The first such result states that if a solution $x(t)$ of the replicator dynamics stays a positive distance away from the boundary of the simplex, then the time-averaged orbit $\bar{x}(t) = t^{-1} \int_0^t x(s) ds$ converges to the set of Nash equilibria of the underlying game (Schuster et al., 1981). The class of games to which this result applies includes zero-sum games (cf. Proposition 6.5) and games satisfying sufficient conditions for permanence (cf. Proposition 6.7 below). The following proposition shows that this convergence property extends to all Hessian dynamics:

Proposition 6.6. *Let \mathcal{G} be a random matching game and let $x(t)$ be a solution orbit of (HD). If $x(t)$ is contained in a compact subset of \mathcal{X}° , the time-averaged orbit $\bar{x}(t) = t^{-1} \int_0^t x(s) ds$ converges to the set of Nash equilibria of \mathcal{G} .*

In the case of the replicator dynamics, this is proved by introducing the auxiliary variables $y_\alpha = \log x_\alpha$ and using the fact that $\dot{y} = \dot{x}/x$. To extend this proof to (HD), we instead define y via the Bregman divergence of h :

Proof of Proposition 6.6. Let $y_\alpha = D_h(e_\alpha, x)$, so $\dot{y}_\alpha = \langle v(x) | x - e_\alpha \rangle = \langle v(x) | x \rangle - v_\alpha(x)$ by Lemma 6.2. Then, for all $\alpha, \beta \in \mathcal{A}$, we get

$$y_\alpha(t) - y_\beta(t) = c_{\alpha\beta} + \int_0^t [v_\beta(x(s)) - v_\alpha(x(s))] ds, \quad (6.12)$$

where $c_{\alpha\beta} = y_\alpha(0) - y_\beta(0)$. Since $x(t)$ is contained in a compact subset of \mathcal{X}° , Lemma 6.1 implies that $\sup_t y_\alpha(t) < \infty$ for all $\alpha \in \mathcal{A}$. Thus, dividing both sides of (6.12) by t and taking the limit $t \rightarrow \infty$, we obtain

$$\lim_{t \rightarrow \infty} [v_\alpha(\bar{x}(t)) - v_\beta(\bar{x}(t))] = 0, \quad (6.13)$$

where we have used the linearity of $v_\alpha(x) = \sum_\beta A_{\alpha\beta}x_\beta$ in x to bring the integral into the arguments of v_α and v_β .

Equation (6.13) implies that if \bar{x}^* is an ω -limit point of $\bar{x}(t)$, then $v_\alpha(\bar{x}^*) = v_\beta(\bar{x}^*)$ for all $\alpha, \beta \in \mathcal{A}$, so \bar{x}^* is a Nash equilibrium of \mathcal{G} . Since \mathcal{X} is compact, every solution of (HD) converges to its ω -limit set, and our assertion follows. \square

Proposition 6.6 applies when the population share of each strategy remains bounded away from zero along all interior solution trajectories, a property known as *permanence*. Formally, a dynamical system on \mathcal{X} is called *permanent* if there exists a threshold $\delta > 0$ such that every interior solution satisfies $\liminf_{t \rightarrow \infty} x_\alpha(t) \geq \delta$ for all $\alpha \in \mathcal{A}$.

Hofbauer and Sigmund (1998) establish a sufficient condition for permanence under the replicator dynamics. Proposition 6.7 extends this result to all continuous Hessian dynamics, providing a sufficient condition for Proposition 6.6 to apply:

Proposition 6.7. *Let \mathcal{G} be a random matching game. Assume that the dynamics (HD) are continuous and there exists some $p \in \mathcal{X}^\circ$ such that*

$$\langle v(x^*)|p - x^* \rangle > 0 \quad \text{for all boundary rest points } x^* \text{ of (HD)}. \quad (6.14)$$

Then, the dynamics (HD) are permanent.

The proof of Proposition 6.7 follows the proof technique of Theorem 13.6.1 of Hofbauer and Sigmund (1998), and is presented in Appendix D.

6.4. Dominated strategies. We conclude by considering the elimination and survival of dominated strategies under (HD). To that end, recall that $\alpha \in \mathcal{A}$ is *strictly dominated* by $\beta \in \mathcal{A}$ if $v_\alpha(x) < v_\beta(x)$ for all $x \in \mathcal{X}$. More generally, $p \in \mathcal{X}$ is *strictly dominated* by $q \in \mathcal{X}$ if $\langle v(x)|p \rangle < \langle v(x)|q \rangle$ for all $x \in \mathcal{X}$, meaning that the average payoff of a small influx of mutants is always higher when the mutants are distributed according to q rather than p . We then say that $p \in \mathcal{X}$ *becomes extinct* along $x(t)$ if $\min\{x_\alpha(t) : \alpha \in \text{supp}(x^*)\} \rightarrow 0$ as $t \rightarrow \infty$ – or equivalently, if there are no ω -limit points of $x(t)$ in $\mathcal{D}(p)$.

Under the replicator dynamics, it is well known that dominated strategies become extinct along every interior solution trajectory (Akin, 1980). As we show below, this elimination result extends to all continuous Hessian dynamics (HD):

Proposition 6.8. *Under all continuous Hessian dynamics (HD), strictly dominated strategies become extinct along every interior solution orbit.*

Proof. The proof follows a standard argument for the replicator dynamics, replacing the KL divergence (6.4b) with the Bregman divergence (6.1). Specifically, Lemma 6.2 implies that along any interior solution $x(t)$,

$$\begin{aligned} \frac{d}{dt} (D_h(p, x) - D_h(q, x)) &= \langle \dot{x}, x - p \rangle_x - \langle \dot{x}, x - q \rangle_x \\ &= \langle \Pi_x(v^\sharp(x)), q - p \rangle_x = \langle v^\sharp(x), q - p \rangle_x = \langle v(x)|q - p \rangle, \end{aligned} \quad (6.15)$$

where the penultimate equality uses the fact that $x \in \mathcal{X}^\circ$. Since q strictly dominates p and v is continuous, we have $\langle v(x)|q - p \rangle \geq a$ for some positive constant $a > 0$, implying that $D_h(p, x(t)) \rightarrow \infty$. Hence, by Lemma 6.1, we conclude that $x(t)$ has no ω -limit points in $\mathcal{D}(p)$. \square

In general, the conclusion of [Proposition 6.8](#) is false for discontinuous Hessian dynamics: [Sandholm et al. \(2008\)](#) construct a four-strategy game with a strictly dominated strategy that is played recurrently by a nonnegligible fraction of the population under the Euclidean projection dynamics (PD). The argument above shows that this strategy must become less common when the state is in the interior of \mathcal{X} ; however, solutions to (PD) are able to enter and leave the boundary of \mathcal{X} , and while there, dominated strategies may become more common. We conjecture that this construction can be suitably extended to all discontinuous Hessian dynamics, but we do not tackle this issue here.

7. LINKS WITH REINFORCEMENT LEARNING

Under a variety of reinforcement learning processes for normal form games, mixed strategies evolve according to the replicator dynamics – see e.g. [Börgers and Sarin \(1997\)](#), [Posch \(1997\)](#), [Rustichini \(1999\)](#), [Hopkins \(2002\)](#) and [Hofbauer et al. \(2009\)](#). We conclude the paper by describing a broader connection between reinforcement learning and Hessian game dynamics.

7.1. Reinforcement learning. Our starting point is a class of reinforcement learning dynamics for N -player normal form games introduced by [Coucheney et al. \(2015\)](#) and [Mertikopoulos and Sandholm \(2016\)](#). Over the course of play, each player maintains a *score vector* representing the cumulative payoffs of each of his strategies; then, at each moment in time, the player selects a mixed strategy by applying a *choice map* to this score vector, similar in function to the perturbed best response maps used in stochastic fictitious play and perturbed best response dynamics ([Fudenberg and Levine, 1998](#); [Hofbauer and Sandholm, 2002, 2007](#)).

Formally, let $v_{k\alpha}(x)$ denote the expected payoff of the α -th strategy of player k at mixed strategy profile $x = (x_1, \dots, x_N)$ in an N -player normal form game. The *choice map* Q_k of player k is then defined as

$$Q_k(y_k) = \arg \max_{x_k \in \mathcal{X}_k} \{\langle y_k | x_k \rangle - h_k(x_k)\}, \quad (7.1)$$

where $\mathcal{X}_k \equiv \Delta(\mathcal{A}_k)$ denotes the mixed strategy space of player k (\mathcal{A}_k being the corresponding strategy set), and $h_k: \mathcal{X}_k \rightarrow \mathbb{R}$ is a smooth, strongly convex *penalty function*. The reinforcement learning process described above can then be written as

$$\begin{aligned} \dot{y}_k &= v_k(x) \\ x_k &= Q_k(y_k). \end{aligned} \quad (\text{RL})$$

Now, let $g_k = \text{Hess } h_k$ and write $n_{k\alpha}(x) = \sum_{\beta \in \mathcal{A}_k} g_{k,\alpha\beta}^{-1}(x)$. [Mertikopoulos and Sandholm \(2016\)](#) showed that when the mixed strategy profile $x(t)$ is interior, its evolution under (RL) is given by

$$\dot{x}_{k\alpha} = \sum_{\beta \in \mathcal{A}_k} \left[g_{k,\alpha\beta}^{-1}(x) - \frac{n_{k\alpha}(x)n_{k\beta}(x)}{\sum_{\gamma} n_{k\gamma}(x)} \right] v_{k\beta}(x). \quad (\text{RLD})$$

A comparison with [\(3.28b\)](#) shows that the dynamics of mixed strategies under (RL) agree with the Hessian dynamics (HD) at interior states.²²

²²While we have defined Riemannian game dynamics for single population games and reinforcement learning for N -person normal form games, this difference is of no consequence. One can similarly define Riemannian game dynamics for multipopulation games, or a symmetrized reinforcement learning process for symmetric two-player normal form games (cf. [Appendix A.2](#)).

7.2. A common derivation of (HD) and (RLD). The derivation of (HD) here and of (RLD) in Mertikopoulos and Sandholm (2016) have very different starting points, leaving the reasons behind their equivalence somewhat mysterious. We now make these reasons clearer by deriving both dynamics using a common set of tools. Here we present the basic idea behind the argument, using the theory of convex duality to establish the equivalence of versions of (HD) of (RLD) defined on the entire positive orthant \mathcal{K}° .²³ Establishing the equivalence of the original processes on \mathcal{X}° using this approach requires further ideas, which we present in Appendix A.2.

The starting point for both (HD) and (RLD) is the potential function h , assumed here to be smooth, strongly convex, and *steep* at the boundary of \mathcal{K} in the sense of (6.5). The *convex conjugate* of h is then defined as²⁴

$$h^*(y) = \sup_{x \in \mathcal{K}^\circ} \{\langle y|x \rangle - h(x)\}, \quad y \in (\mathbb{R}^A)^*. \quad (7.2)$$

Since h is steep and strongly convex, the supremum above is attained at a unique point $Q_{\mathcal{K}}(y) \in \mathcal{K}^\circ$ (Rockafellar, 1970, Theorem 26.5).²⁵ By the first-order optimality conditions for (7.2), we then get

$$Q_{\mathcal{K}}(y) \equiv \arg \max_{x \in \mathcal{K}^\circ} \{\langle y|x \rangle - h(x)\} = (dh)^{-1}(y), \quad (7.3)$$

where $(dh)^{-1}$ is the inverse function of dh . Applying the envelope theorem to (7.2), we then obtain

$$dh^*(y) = Q_{\mathcal{K}}(y). \quad (7.4)$$

It follows from (7.3) and (7.4) that $dh(dh^*(y)) \equiv y$, so differentiating and rearranging yields

$$\text{Hess } h^*(y) = (\text{Hess } h(x))^{-1} \quad (7.5)$$

with the inverse of the Hessian matrix of h being evaluated at $x = Q_{\mathcal{K}}(y)$.

We now introduce and compare full-dimensional analogues of continuous Hessian dynamics and (symmetric) reinforcement learning, taking the orthant \mathcal{K}° as the state space (and so defining payoffs v on \mathcal{K}°). In the case of (HD), the full-dimensional domain makes projections redundant, so the induced dynamics take the form

$$\dot{x} = v^\sharp(x) = g^{-1}(x) v(x)^\top = (\text{Hess } h(x))^{-1} v(x)^\top. \quad (\text{HD}_{\mathcal{K}})$$

For its part, the full-dimensional analogue of (RL) is

$$\begin{aligned} \dot{y} &= v(x), \\ x &= Q_{\mathcal{K}}(y), \end{aligned} \quad (\text{RL}_{\mathcal{K}})$$

with $Q_{\mathcal{K}}$ defined as in (7.3) above. Differentiating (RL_K) and applying (7.4) and (7.5) then yields

$$\dot{x} = dQ_{\mathcal{K}}(y) v(x)^\top = \text{Hess } h^*(y) v(x)^\top = (\text{Hess } h(x))^{-1} v(x)^\top. \quad (\text{RLD}_{\mathcal{K}})$$

We can express this argument in words. By definition, (full-dimensional) Riemannian dynamics are obtained by transforming payoffs at an interior population

²³The usefulness of convex conjugates in analyzing maps of the form (7.1) is well known in learning and optimization – see e.g. Nemirovski and Yudin (1983), Hofbauer and Sandholm (2002), Shalev-Shwartz (2011), and Kwon and Mertikopoulos (2014).

²⁴For a comprehensive treatment, see Rockafellar, 1970 or Hiriart-Urruty and Lemaréchal, 2001.

²⁵Since h is strongly convex, it is bounded below by a strictly convex quadratic function (Hiriart-Urruty and Lemaréchal, 2001, Theorem B.4.1.1), which in turn implies that the domain of h^* is $(\mathbb{R}^n)^*$ (Rockafellar, 1970, Corollary 13.1).

state x using the matrix $g^{-1}(x)$. In the Hessian case, the metric g admits a potential function h , so we have $g^{-1}(x) = (\text{Hess } h(x))^{-1}$ by definition. As for reinforcement learning, differentiation shows that the dynamics of the mixed strategy x are obtained by transforming payoffs at x by the derivative matrix $dQ_{\mathcal{K}}(y)$ of the choice map (7.3). Basic facts about convex conjugacy imply that this derivative is equal to $(\text{Hess } h(x))^{-1}$, where h is the penalty function that generates $Q_{\mathcal{K}}$. This argument establishes the equivalence of $(\text{HD}_{\mathcal{K}})$ and $(\text{RLD}_{\mathcal{K}})$. For a corresponding argument for the original processes (HD) and (RLD) , see [Appendix A.2](#).

7.3. Boundary behavior in the nonsteep regime. Since continuous Hessian dynamics coincide with the reinforcement scheme (RL) when the penalty functions h_k are steep, certain results for interior trajectories – [Propositions 6.6](#) and [6.8](#) in particular – can be obtained directly from the analysis of [Mertikopoulos and Sandholm \(2016\)](#). On the other hand, in the nonsteep regime, (RLD) and (HD) agree at interior states, but their behaviors at the boundary differ in a fundamental way. Specifically, while the boundary behavior of discontinuous Hessian dynamics is defined using closest point projections, the reinforcement learning process (RL) for nonsteep h can no longer be reduced to mixed strategy dynamics at all. Instead one must work explicitly with the score variables y_k , which continue to aggregate payoff data of all strategies, even those that are not used. Among other things, this means that a strong cumulative performance of an unused strategy will return it to use.

This difference between how the processes are defined on the boundary has important consequences. For instance, while [Sandholm et al. \(2008\)](#) show that strictly dominated strategies may survive under the Euclidean projection dynamics, [Mertikopoulos and Sandholm \(2016\)](#) prove that the reinforcement learning process (RL) *always* eliminates dominated strategies, whether the penalty functions are steep or not. Under both processes, dominated strategies are initially eliminated along interior solution trajectories. But while they may resurface under (HD) , the score variables of (RL) continue to register the poor performance of these strategies, ensuring that they remain extinct for all time.

APPENDIX A. CONNECTIONS WITH OTHER GAME DYNAMICS

Throughout this appendix, we write $\mathbf{1} \in \mathbb{R}^n$ for the n -dimensional column vector of ones and $\Phi = I - \frac{1}{n}\mathbf{1}\mathbf{1}^\top$ for the Euclidean orthogonal projection of \mathbb{R}^n onto $\mathbb{R}_0^n = \{z \in \mathbb{R}^n : \mathbf{1}^\top z = 0\} = \text{span}(\mathbf{1})^\perp$.²⁶ Recall also that the (*Moore-Penrose*) *pseudoinverse* of a matrix $M \in \mathbb{R}^{n \times n}$ is the unique matrix $M^+ \in \mathbb{R}^{n \times n}$ such that (i) $M^+y = 0$ whenever $y \in \text{range}(M)^\perp$; and (ii) $M^+y = x$ whenever $x \in \text{ker}(M)^\perp$, $y \in \text{range}(M)$, and $Mx = y$ ([Friedberg et al., 2002, Sec. 6.7](#)). Since a symmetric matrix $M \in \mathbb{R}^{n \times n}$ satisfies $\text{range}(M) = \text{ker}(M)^\perp$, we have the following well-known algebraic characterization of pseudoinverses:

Lemma A.1. *Let $M \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Then, M^+ is the unique matrix that (i) inverts M on $\text{ker}(M)^\perp = \text{range}(M)$; and (ii) satisfies $\text{ker}(M^+) = \text{ker}(M)$.*

²⁶In the above and what follows, W^\perp denotes the orthogonal complement of a subspace W of \mathbb{R}^n , defined with respect to the ordinary Euclidean metric. Even though this might seem to suggest that the Euclidean metric plays a special role in what follows, it is just an artifact of writing everything in coordinates instead of abstractly; for a detailed discussion, see [Lee \(2003\)](#).

A.1. Interior equivalence of Riemannian dynamics and Hopkins' dynamics. We now derive the equivalence between (RmD) and Hopkins' dynamics (2.7) on \mathcal{X}° , as noted in Section 2.3. To begin with, we say that $M \in \mathbb{R}^{n \times n}$ is a *Hopkins matrix* if it is positive-definite with respect to \mathbb{R}_0^n and maps $\mathbf{1}$ to 0. The following lemma establishes a basic characterization of Hopkins matrices:

Lemma A.2.

(i) *If $S \in \mathbb{R}^{n \times n}$ is symmetric positive-definite, then $(\Phi S \Phi)^+$ is a Hopkins matrix and*

$$(\Phi S \Phi)^+ = S^{-1} - \frac{S^{-1} \mathbf{1} \mathbf{1}^\top S^{-1}}{\mathbf{1}^\top S^{-1} \mathbf{1}}. \quad (\text{A.1})$$

(ii) *Conversely, if M is a Hopkins matrix, then $M = (\Phi S \Phi)^+$, where $S = M + \mathbf{1} \mathbf{1}^\top$ is symmetric positive definite.*

Proof. To prove part (i), let S be symmetric positive definite. Then the symmetric matrix $\Phi S \Phi$ is positive definite with respect to \mathbb{R}_0^n and maps $\mathbf{1}$ to 0, so $\text{range}(\Phi S \Phi) = \ker(\Phi S \Phi)^\perp = \mathbb{R}_0^n$. If we denote the right-hand side of (A.1) by \bar{S} , a straightforward calculation shows that $\bar{S} \Phi S \Phi z = z$ for all $z \in \mathbb{R}_0^n$ and $\bar{S} \mathbf{1} = 0$. Thus Lemma A.1 implies that $\bar{S} = (\Phi S \Phi)^+$. That this is a Hopkins matrix is immediate from the fact that $\text{range}(\Phi S \Phi) = \ker(\Phi S \Phi)^\perp = \mathbb{R}_0^n$ and Lemma A.1.

To prove part (ii), let M be a Hopkins matrix and let $S = M + \mathbf{1} \mathbf{1}^\top$. Clearly S is symmetric positive-definite. Moreover, writing out $(\Phi S \Phi)^+$ using the right-hand side of (A.1) and simplifying the result yields $(\Phi S \Phi)^+ = M$. \square

Proposition A.3 below establishes the equivalence between (2.7) and (RmD) on \mathcal{X}° , and provides a concise third representation for both dynamics. Observe that (A.2c) is expression (3.28b) for (RmD) on \mathcal{X}° , but written in matrix form.

Proposition A.3. *Let $\dot{x} = V(x)$ be a dynamical system on \mathcal{X}° . Then, the following are equivalent:*

(i) *There is a smooth field of Hopkins matrices $M: \mathcal{X}^\circ \rightarrow \mathbb{R}^{\mathcal{A} \times \mathcal{A}}$ such that*

$$V(x) = M(x) v(x)^\top \quad \text{for all } x \in \mathcal{X}^\circ. \quad (\text{A.2a})$$

(ii) *There is a smooth field of symmetric positive-definite matrices $H: \mathcal{X}^\circ \rightarrow \mathbb{R}^{\mathcal{A} \times \mathcal{A}}$ such that*

$$V(x) = (\Phi H(x) \Phi)^+ v(x)^\top \quad \text{for all } x \in \mathcal{X}^\circ. \quad (\text{A.2b})$$

(iii) *There is a smooth Riemannian metric g on \mathcal{K}° such that*

$$V(x) = \left(g^{-1}(x) - \frac{g^{-1}(x) \mathbf{1} \mathbf{1}^\top g^{-1}(x)}{\mathbf{1}^\top g^{-1}(x) \mathbf{1}} \right) v(x)^\top \quad \text{for all } x \in \mathcal{X}^\circ. \quad (\text{A.2c})$$

Proof. The equivalence of (i) and (ii) follows from Lemma A.2, and the equivalence of (ii) and (iii) follows from Lemma A.2(i) with $S = H(x) = g(x)$.²⁷ \square

²⁷To formally complete the argument that (ii) implies (iii), we observe without proof that the field S on \mathcal{X}° can be smoothly extended to \mathcal{K}° .

A.2. Continuous Hessian dynamics and reinforcement learning. We now complete the common derivation of (HD) and (RLD) on \mathcal{X}° initiated in Section 7.2. Proposition A.3 above shows that the continuous Hessian dynamics (HD) can be expressed as

$$\dot{x} = (\Phi H(x)\Phi)^+ v(x)^\top. \quad (\text{A.3})$$

We now extend the argument from Section 7.2 to show that the reinforcement learning dynamics (RLD) also take this form.

To that end, let $\mathcal{Z} = \{x - \frac{1}{n}\mathbf{1} : x \in \mathcal{X}^\circ\} \subset \mathbb{R}_0^n$, and define $h_{\mathcal{Z}} : \mathcal{Z} \rightarrow \mathbb{R}$ as

$$h_{\mathcal{Z}}(z) = h(z + \frac{1}{n}\mathbf{1}). \quad (\text{A.4})$$

Since \mathcal{Z}° is open relative to \mathbb{R}_0^n , the differential $dh_{\mathcal{Z}}$ of $h_{\mathcal{Z}}$ at $z \in \mathcal{Z}^\circ$ can be represented by a covector $dh_{\mathcal{Z}}(z)$ in $(\mathbb{R}_0^n)^*$. Imposing the Euclidean metric on \mathbb{R}_0^n for convenience, we can identify $(\mathbb{R}_0^n)^*$ with the set of covectors whose components sum to zero, and write the differential and Hessian of $h_{\mathcal{Z}}$ at $z \in \mathcal{Z}^\circ$ as

$$dh_{\mathcal{Z}}(z) = dh(z + \frac{1}{n}\mathbf{1})\Phi, \quad (\text{A.5})$$

$$\text{Hess } h_{\mathcal{Z}}(z) = \Phi H(z + \frac{1}{n}\mathbf{1})\Phi. \quad (\text{A.6})$$

The convex conjugate $h_{\mathcal{Z}}^*$ of $h_{\mathcal{Z}}$ is then defined as

$$h_{\mathcal{Z}}^*(y_0) = \max_{z \in \mathcal{Z}} \{ \langle y_0 | z \rangle - h_{\mathcal{Z}}(z) \}, \quad y_0 \in (\mathbb{R}_0^n)^*, \quad (\text{A.7})$$

where the fact that the domain of $h_{\mathcal{Z}}^*$ is all of $(\mathbb{R}_0^n)^*$ follows from the compactness of \mathcal{Z} . By the basics of convex conjugation (Rockafellar, 1970, Chapter 26), the maps $dh_{\mathcal{Z}}^* : (\mathbb{R}_0^n)^* \rightarrow \mathcal{Z}$ and $dh_{\mathcal{Z}} : \mathcal{Z} \rightarrow (\mathbb{R}_0^n)^*$ are inverse to one another, so we have

$$\text{Hess } h_{\mathcal{Z}}^*(y_0) = (\text{Hess } h_{\mathcal{Z}}(dh_{\mathcal{Z}}^*(y_0)))^{-1}, \quad (\text{A.8})$$

with both sides of the equality in (A.8) understood as linear maps from \mathbb{R}_0^n to itself.

Now recall that the (symmetric) reinforcement learning process is defined by

$$\begin{aligned} \dot{y} &= v(x) \\ x &= Q(y). \end{aligned} \quad (\text{RL})$$

The choice map Q , defined in (7.1) in terms of h , satisfies $Q(y) = Q(y\Phi)$ for all $y \in \mathbb{R}^n$. Thus, applying definition (A.4), we can express Q in terms of $h_{\mathcal{Z}}$ as

$$Q(y) = \arg \max_{x \in \mathcal{X}} \{ y\Phi x - h(x) \} = \arg \max_{z \in \mathcal{Z}} \{ y\Phi z - h_{\mathcal{Z}}(z) \} + \frac{1}{n}\mathbf{1} = dh_{\mathcal{Z}}^*(y\Phi) + \frac{1}{n}\mathbf{1}. \quad (\text{A.9})$$

Hence, substituting (A.9) into (RL), differentiating, and using Eqs. (RL), (A.6), (A.8) and (A.9) yields

$$\begin{aligned} \dot{x} &= \text{Hess } h_{\mathcal{Z}}^*(v(x)\Phi) \Phi v(x)^\top = (\text{Hess } h_{\mathcal{Z}}(dh_{\mathcal{Z}}^*(v(x)\Phi)))^{-1} \Phi v(x)^\top \\ &= (\text{Hess } h_{\mathcal{Z}}(x - \frac{1}{n}\mathbf{1}))^{-1} \Phi v(x)^\top = (\Phi H(x)\Phi)^+ v(x)^\top, \end{aligned} \quad (\text{A.10})$$

as specified in (A.3).

APPENDIX B. EXTENSIONS OF RIEMANNIAN METRICS

Here we present some technical results concerning the extension of Riemannian metrics from \mathcal{K}° to \mathcal{K} . Proposition B.1 shows that an extendable metric g on \mathcal{K} induces a well-defined scalar product at all points of \mathcal{K} :

Proposition B.1. *Let g be an extendable Riemannian metric on \mathcal{K} . Then, for all $x \in \mathcal{K}$, there exists a unique scalar product $\langle \cdot, \cdot \rangle_x$ on $\text{dom } g(x)$ such that $\langle w, w' \rangle_{x_k} \rightarrow \langle w, w' \rangle_x$ for all $w, w' \in \text{dom } g(x)$ and for every interior sequence $x_k \rightarrow x$. Moreover, if g is minimal-rank extendable, we have $g_{\alpha\beta}^\sharp(x) = 0$ whenever $\alpha, \beta \notin \text{supp}(x)$.*

Proof. Fix some $x \in \mathcal{K}$ and write $g^\sharp(x) = Q^\top \Lambda Q$ where the diagonal matrix Λ consists of the eigenvalues of $g^\sharp(x)$ and Q is an orthogonal matrix ($Q^\top = Q^{-1}$) whose columns are the eigenvectors of $g^\sharp(x)$. Since $g^\sharp(x)$ is positive-semidefinite, its eigenvalues are nonnegative. Furthermore, since $\text{dom } g(x) = \text{im } g^\sharp(x)$, every eigenvector of a nonzero eigenvalue of $g^\sharp(x)$ must lie in $\text{dom } g(x)$: indeed, if $g^\sharp(x)z = \lambda z$ for some $\lambda > 0$, we will also have $z = g^\sharp(x)z\lambda^{-1}$, i.e. $z \in \text{im } g^\sharp(x) = \text{dom } g(x)$. As a result, we may write $g^\sharp(x) = \sum_{\lambda > 0} \lambda u_\lambda u_\lambda^\top$ where the summation is taken over all positive eigenvalues $\lambda > 0$ of $g^\sharp(x)$ (assumed for convenience to be distinct) and u_λ is the corresponding column of Q . The metric tensor of the induced scalar product $\langle \cdot, \cdot \rangle_x$ at x is then defined as the pseudoinverse $(g^\sharp(x))^+$ of $g(x)$ (see [Appendix A](#)), given here by

$$(g^\sharp(x))^+ = \sum_{\lambda > 0} \lambda^{-1} u_\lambda u_\lambda^\top. \quad (\text{B.1})$$

Our continuity and uniqueness claims are then immediate.

Finally, to show that $g_{\alpha\beta}^\sharp(x) = 0$ if $\alpha \notin \text{supp}(x)$ and g is minimal-rank extendable, simply note that $g_{\alpha\beta}^\sharp(x) = e_\alpha^\top g^\sharp(x) e_\beta = \sum_\lambda \lambda e_\alpha^\top u_\lambda u_\lambda^\top e_\beta = 0$ because all eigenvectors of $g^\sharp(x)$ with positive eigenvalues lie in $\mathbb{R}^{\text{supp}(x)} = \text{dom } g(x)$. \square

Remark B.1. In the minimal-rank case, the scalar product $\langle \cdot, \cdot \rangle_x$ on $\text{dom } g(x)$ can be represented by the matrix $g(x) = (g^\sharp(x))^+$. This means that the submatrix $(g_{\alpha\beta}(x))_{\alpha, \beta \in \text{supp}(x)}$ is the inverse of the submatrix $(g_{\alpha\beta}^\sharp(x))_{\alpha, \beta \in \text{supp}(x)}$ while the remaining components (which have no geometric significance) are set to 0.

Next we establish the correctness of the formulas (3.28) for dynamics generated by a minimal-rank extendable metrics.

Proposition B.2. *The formulas (3.28) describe (RmD) generated by a minimal-rank extendable metric g at all states $x \in \mathcal{X}$; moreover, the sums in (3.28) need only be taken over $\beta \in \text{supp}(x)$.*

Proof. The case $x \in \mathcal{X}^\circ$ is covered in the text. Otherwise, if $x \in \text{bd}(\mathcal{X})$, the corresponding g -admissible set is

$$\text{Adm}_g(x) = \text{TC}_{\mathcal{X}}(x) \cap \text{T}_{\mathcal{K}}(x) = \mathbb{R}_0^{\mathcal{A}} \cap \mathbb{R}^{\text{supp}(x)} = \text{T}_{\mathcal{X}}(x). \quad (\text{B.2})$$

By definition, $v^\sharp(x)$ and $n(x)$ lie in $\text{im } g^\sharp(x) = \mathbb{R}^{\text{supp}(x)}$. Moreover, $n(x)$ is normal to $\text{T}_{\mathcal{X}}(x) = \mathbb{R}_0^{\mathcal{A}} \cap \mathbb{R}^{\text{supp}(x)}$ because

$$\langle n(x), z \rangle_x = \mathbb{1} g^\sharp(x) g(x) z = \langle \mathbb{1} | z \rangle = 0 \quad \text{for all } z \in \mathbb{R}_0^{\mathcal{A}}, \quad (\text{B.3})$$

where the second equality follows from [Remark B.1](#).

Eq. (3.36) shows that the right-hand side of (RmD) is equal to $\Pi_x(v^\sharp(x))$. In light of the facts above, (3.35) shows that $\Pi_x(v^\sharp(x))$ is equal to the right-hand side of (3.28a), the novelty being that $v_\alpha^\sharp(x)$ and $n_\alpha(x)$ vanish whenever $\alpha \notin \text{supp}(x)$. These claims prove the first statement in the proposition. To establish the second statement, simply observe that $g_{\alpha\beta}^\sharp(x) = 0$ whenever α or β is not in $\text{supp}(x)$ by [Proposition B.1](#), so including $\beta \notin \text{supp}(x)$ in the sums in (3.28) is irrelevant. \square

APPENDIX C. CONVERGENCE AND STABILITY IN DYNAMICAL SYSTEMS

C.1. Definitions. Throughout this appendix, we focus on the dynamics

$$\dot{x} = V(x), \quad x \in \mathcal{X}, \quad (\text{D})$$

and we assume that they admit unique solutions from every initial condition. With this in mind, we say that x^* is an ω -*limit point* of the solution orbit $x(t)$ if there is an increasing sequence of times $t_n \uparrow \infty$ such that $x(t_n) \rightarrow x^*$. We further say that x^* is *Lyapunov stable* if, for every neighborhood U of x^* , there exists a neighborhood U' of x^* such that every solution orbit $x(t)$ that starts in U' is contained in U for all $t \geq 0$. Finally, we say that x^* is *attracting* if there is a neighborhood U of x^* such that every solution that starts in U converges to x^* , and x^* is called *asymptotically stable* if it is Lyapunov stable and attracting. In this case, the maximal (relatively) open set of states from which solutions converge to x^* is called the *basin of x^** ; if the basin of x^* is all of \mathcal{X} , we say that x^* is *globally asymptotically stable*.

C.2. A global convergence result. A standard result from dynamical systems states that if a smooth dynamical system on a compact set admits a strict global Lyapunov function, all ω -limit points are rest points (see e.g. Sandholm, 2010b, Theorem 7.B.3). The proof of this result relies on the continuity of solutions on initial conditions, a property which is not easily established for discontinuous dynamics. In [Proposition C.1](#) below, we present a global convergence result that does not require continuity of solutions in initial conditions, but instead relies on a l.s.c. lower bound on the derivative of the Lyapunov function. To state it, let

$$\text{RP} = \{x \in \mathcal{X} : V(x) = 0\} \quad (\text{C.1})$$

denote the set of rest points of the dynamics (D). We then have:

Proposition C.1. *Let $x(t)$ be an absolutely continuous solution orbit of (D) and let $\Gamma_+ = x(\mathbb{R}_+)$ denote the set of points visited by $x(t)$. Assume further that RP is closed and there exist functions $L: \mathcal{X} \rightarrow \mathbb{R}$ and $\phi: \mathcal{X} \rightarrow \mathbb{R}_+$ such that*

- (i) L is differentiable in a neighborhood of Γ_+ .
- (ii) ϕ is lower semi-continuous and $\phi(x) = 0$ if and only if $x \in \text{RP}$.
- (iii) $\langle dL(x)|V(x) \rangle \geq \phi(x)$ for all $x \in \Gamma_+$.

Then, $x(t)$ converges to RP.

Proof. By absolute continuity and Conditions (i) and (iii) above, we get

$$L(x(t)) - L(x(0)) = \int_0^t \langle dL(x(s))|V(x(s)) \rangle ds \geq \int_0^t \phi(x(s)) ds \geq 0, \quad (\text{C.2})$$

i.e. L is nondecreasing along $x(t)$. Furthermore, since \mathcal{X} is compact, $x(t)$ admits at least one ω -limit point $x^* \in \text{cl}(\Gamma_+) \subseteq \mathcal{X}$. Assume now that $x(t)$ admits an ω -limit point x^ω such that $V(x^\omega) \neq 0$. Since RP is closed, Condition (ii) implies that there is a compact neighborhood K of x^ω and some $a > 0$ such that $\phi(x) \geq a > 0$ for all $x \in K$. With this in mind, we consider two complementary cases below:

Case 1. Suppose there exists some $T \geq 0$ such that $x(t) \in K$ for all $t \geq T$. Then $\phi(x(t)) \geq a$ for all $t \geq T$, so (C.2) yields $\lim_{t \rightarrow \infty} L(x(t)) = \infty$, a contradiction.

Case 2. Assume instead that, for all $T \geq 0$, we have $x(t) \notin K$ for some $t \geq T$. In this case, there exist open neighborhoods U and U' of x^ω with $\text{cl}(U) \subseteq U' \subset K$, and interlaced sequences $t_n, t'_n \uparrow \infty$ such that, for all n : (i) $t_n < t'_n < t_{n+1}$; (ii) $x(t_n) \in U$, $x(t'_n) \in K \setminus U'$; and (iii) $x(t) \in K$ whenever $t \in [t_n, t'_n]$. Then, since $|\dot{x}_\alpha(t)|$ is bounded from above by $V_{\max} \equiv \sup_{x \in \mathcal{X}} \max_\beta |V_\beta(x)| < \infty$, the time intervals $\delta_n \equiv t'_n - t_n$ will be bounded from below by $\delta_{\min} \equiv \text{dist}(\text{cl}(U), K \setminus U')/V_{\max} > 0$. We thus get

$$L(x(t'_n)) - L(x(0)) \geq \int_0^{t'_n} \phi(x(s)) ds \geq \sum_{j=1}^n \int_{t_j}^{t'_j} \phi(x(s)) ds \geq an\delta_{\min}, \quad (\text{C.3})$$

i.e. $L(x(t'_n)) \rightarrow \infty$, a contradiction. \square

To apply [Proposition C.1](#) to discontinuous Riemannian dynamics in potential games, we need the following result:

Lemma C.2. *The speed of motion $\|V(x)\|_x$ of the dynamics (RmD) is l.s.c. on \mathcal{X} .*

Proof. If the dynamics (RmD) are continuous, our claim follows immediately from the continuity of the underlying metric – in fact, $\|V(x)\|_x$ is continuous in this case. Otherwise, if (RmD) is discontinuous, recall that $V(x) \equiv \Pi_x(v^\sharp(x))$ is simply the projection of $v^\sharp(x)$ on the tangent cone $\text{TC}_\mathcal{X}(x)$ to \mathcal{X} at x (because $\text{Adm}_g(x) = \text{TC}_\mathcal{X}(x)$ in that case). Therefore, if we write $V^\perp(x) = v^\sharp(x) - V(x)$ for the projection of $v^\sharp(x)$ on the normal cone $\text{NC}_\mathcal{X}(x)$ to \mathcal{X} at x , Moreau's decomposition theorem and the Cauchy-Schwarz inequality yield

$$\langle v^\sharp(x), z \rangle_x = \langle V(x) + V^\perp(x), z \rangle_x = \langle V(x), z \rangle_x \leq \|V(x)\|_x \|z\|_x, \quad (\text{C.4})$$

for all $z \in \text{TC}_\mathcal{X}(x)$, with the inequality binding if and only if $z \propto V(x)$. We thus obtain the characterization

$$\|V(x)\|_x = \max_{z \in \text{TC}_\mathcal{X}(x) \cap B(x)} \langle v^\sharp(x), z \rangle_x, \quad (\text{C.5})$$

where $B(x) = \{z \in \mathbb{R}^A : \|z\|_x \leq 1\}$.

Note now that the correspondence $x \mapsto \text{TC}_\mathcal{X}(x)$ is l.s.c. because it is constant on the interior of each face of \mathcal{X} and $\text{TC}_\mathcal{X}(x) \subseteq \text{TC}_\mathcal{X}(y)$ whenever $\text{supp}(x) \subseteq \text{supp}(y)$. This shows that the constraint correspondence $x \mapsto \text{TC}_\mathcal{X}(x) \cap B(x)$ of (C.5) is l.s.c.; since the objective function $\langle v^\sharp(x), z \rangle_x$ of (C.5) is jointly continuous in x and z , a precursor to the maximum theorem ([Aliprantis and Border, 1999, Lemma 16.30](#)) implies that $x \mapsto \|V(x)\|_x$ is itself l.s.c., as claimed. \square

APPENDIX D. ADDITIONAL PROOFS

In this appendix, we collect some proofs that are too technical for the main text. We begin with the proof of [Proposition 5.1](#) regarding the existence and uniqueness of solutions to (RmD). As noted in [Section 5](#), we only need to prove part (ii), which concerns the case of full-rank extendable metrics. Existence of forward solutions of (RmD) on \mathcal{X} follows from general results of [Aubin and Cellina \(1984\)](#) on solutions to discontinuous differential equations; see [Lahkar and Sandholm \(2008\)](#) for a summary of their argument. Thus, it remains to show that forward solutions to (RmD) from each initial condition in \mathcal{X} are unique. This conclusion follows from the following lemma:

Lemma D.1. *Let $x(t)$ and $x'(t)$ be solutions to (RmD), and let*

$$P(t) = \|x'(t) - x(t)\|_{x(t)}^2 e^{-\lambda t}. \quad (\text{D.1})$$

If $\lambda > 0$ is large enough, then $P(t)$ is nonincreasing.

Given this lemma, we immediately obtain:

Proof of Proposition 5.1. Let $x(t)$ and $x'(t)$ be solutions of (RmD) with $x(0) = x'(0)$. We then get $P(t) = P(0) = 0$ for all $t \geq 0$, so $x(t) = x'(t)$ for all $t \geq 0$. \square

To prove Lemma D.1, we need one final auxiliary result. Let $V(x) = \Pi_x(v^\sharp(x))$ denote the right-hand side of (RmD). Then, as we show below, V satisfies a one-sided Lipschitz condition with respect to the underlying metric:

Lemma D.2. *There exists some $K_V > 0$ such that*

$$\langle V(x') - V(x), x' - x \rangle_x \leq K_V \|x' - x\|_x^2 \quad \text{for all } x, x' \in \mathcal{X}. \quad (\text{D.2})$$

Proof of Lemma D.2. Write $w(x) = v^\sharp(x)$ and $w^\perp(x) = w(x) - \Pi_x(w(x))$. Since g is full-rank extendable, Π_x is the orthogonal projection onto $\text{TC}_{\mathcal{X}}(x)$ with respect to $\langle \cdot, \cdot \rangle_x$. We thus obtain

$$\begin{aligned} \langle V(x') - V(x), x' - x \rangle_x &= \langle w(x') - w(x), x' - x \rangle_x - \langle w^\perp(x') - w^\perp(x), x' - x \rangle_x \\ &= \langle w(x') - w(x), x' - x \rangle_x + \langle w^\perp(x), x' - x \rangle_x \\ &\quad + \langle w^\perp(x'), x - x' \rangle_{x'} + (w^\perp(x'))^\top (g(x') - g(x))(x' - x) \\ &\leq K_w \|x' - x\|_x^2 + (w^\perp(x'))^\top (g(x') - g(x))(x' - x), \end{aligned} \quad (\text{D.3})$$

where the bound for the first term in the last line follows from the Cauchy-Schwarz inequality and the Lipschitz continuity of w , while the rest follows from Moreau's decomposition theorem. To bound the last term, write $g_\alpha(x)$ for the α -th row of $g(x)$, let $W_{\max}^\perp = \max_{\alpha \in \mathcal{A}} \max_{x \in \mathcal{X}} w_\alpha^\perp(x)$, and let $\|\cdot\|_2$ denote the standard Euclidean norm. Then, if $C > 0$ is chosen sufficiently large, we get

$$\begin{aligned} &(w^\perp(x'))^\top (g(x') - g(x))(x' - x) \\ &\leq \sum_{\alpha \in \mathcal{A}} w_\alpha^\perp(x') \|g_\alpha(x') - g_\alpha(x)\|_2 \|x' - x\|_2 \\ &\leq W_{\max}^\perp \|x' - x\|_2 \sum_{\alpha \in \mathcal{A}} \|g_\alpha(x') - g_\alpha(x)\|_2 \leq W_{\max}^\perp C \|x' - x\|_x^2. \end{aligned} \quad (\text{D.4})$$

In the above, the first inequality is an immediate corollary of the Cauchy-Schwarz inequality; the last one follows from the equivalence of norms on $\mathbb{R}^{\mathcal{A}}$ and the fact that g is C^1 on \mathcal{X} ; finally, C can be chosen independently of x and x' because \mathcal{X} is compact. Combining (D.3) and (D.4) completes our proof. \square

With Lemma D.2 at hand, we finally obtain:

Proof of Lemma D.1. Define $\dot{g}(x) = (\dot{g}_{\alpha\beta}(x))_{\alpha,\beta \in \mathcal{A}}$ by $\dot{g}_{\alpha\beta}(x) = \langle dg_{\alpha\beta}(x) | V(x) \rangle = \sum_{\kappa \in \mathcal{A}} V_\kappa(x) \partial_\kappa g_{\alpha\beta}(x)$, and let $K_g = \max_{\alpha,\beta \in \mathcal{A}} \max_{x \in \mathcal{X}} \dot{g}_{\alpha\beta}(x) < \infty$ (recall that g is C^1). Then, for all $t \geq 0$ such that $x(t)$ and $x'(t)$ are differentiable, we have

$$\begin{aligned} \dot{P} &= 2 \langle V(x') - V(x), x' - x \rangle_x e^{-\lambda t} + (x' - x)^\top \dot{g}(x)(x' - x) e^{-\lambda t} - \lambda \|x' - x\|_x^2 e^{-\lambda t} \\ &\leq -(\lambda - 2K_V - K_g) \|x' - x\|_x^2 e^{-\lambda t}, \end{aligned} \quad (\text{D.5})$$

where we used [Lemma D.2](#) to bound the second term in the first line. Taking $\lambda > 2K_V + K_g$ then yields $\dot{P} \leq 0$; since $x(t)$ and $x'(t)$ are absolutely continuous, we conclude that $P(t)$ is nondecreasing. \square

We close this appendix with the proof of our permanence criterion:

Proof of [Proposition 6.7](#). Define $P: \mathcal{K} \rightarrow \mathbb{R}$ as $P(x) = -\exp(\sum_{\alpha} p_{\alpha} D_h(e_{\alpha}, x))$ for $x \in \mathcal{K}^{\circ}$ and $P(x) = 0$ for $x \in \text{bd}(\mathcal{K})$. The steepness of h implies that P is continuous, while [Lemma 6.2](#) implies that $\frac{d}{dt} \log(P(x)) = \Psi(x) \equiv \langle v(x) | p - x \rangle$ for all $x \in \mathcal{X}^{\circ}$. Hence, by [Theorem 12.2.1 of Hofbauer and Sigmund \(1998\)](#), it suffices to show that the function Ψ is an average Lyapunov function for [\(HD\)](#), meaning that, for every initial condition $x(0) \in \text{bd}(\mathcal{X})$, there is a $t > 0$ such that

$$\frac{1}{t} \int_0^t \Psi(x(s)) ds > 0. \quad (\text{D.6})$$

We proceed by induction on the cardinality of the support of the initial condition. The claim is trivial if this cardinality is 1. For the inductive step, suppose that [\(D.6\)](#) holds when the cardinality is $k \in \{1, \dots, |\mathcal{A}| - 2\}$, and consider an initial condition $x(0)$ whose support \mathcal{A}' has cardinality $k+1$. If $x(t)$ converges to the boundary of the face \mathcal{X}' of \mathcal{X} spanned by \mathcal{A}' , then our claim follows from the inductive hypothesis and the same arguments as in the proof of [Theorem 12.2.2 in Hofbauer and Sigmund \(1998\)](#). If instead $x(t)$ does not converge to the boundary of \mathcal{X}' , then there exists a $\delta > 0$ and an increasing sequence of times $t_n \uparrow \infty$ with $x_{\alpha}(t_n) \geq \delta > 0$ for all $\alpha \in \mathcal{A}'$. Then, letting $\bar{x}_{\alpha}(t) = t^{-1} \int_0^t x_{\alpha}(s) ds$ and $\bar{u}(t) = t^{-1} \int_0^t \langle v(x(s)) | x(s) \rangle ds$, we may assume (by descending to a subsequence of t_n if necessary) that $\bar{x}(t_n)$ and $\bar{u}(t_n)$ converge to some \bar{x}^* and \bar{u}^* respectively as $n \rightarrow \infty$.

We now claim that $v_{\alpha}(\bar{x}^*) = \bar{u}^*$ for all $\alpha \in \mathcal{A}'$, implying that \bar{x}^* is a restricted equilibrium of \mathcal{G} . Indeed, let $y_{\alpha}(t) = D_h(e_{\alpha}, x(t))$ for all $\alpha \in \mathcal{A}'$. Then, $\dot{y}_{\alpha} = \langle v(x) | x \rangle - v_{\alpha}(x)$ by [Lemma 6.2](#), so the linearity of $v(x)$ in x implies that

$$\frac{1}{t} \int_0^t \langle v(x(s)) | x(s) \rangle ds - v_{\alpha}(\bar{x}(t)) = \frac{y_{\alpha}(t) - y_{\alpha}(0)}{t}. \quad (\text{D.7})$$

Given that $x(t_n)$ remains a minimal positive distance away from $\text{bd}(\mathcal{X}')$, it follows that $y_{\alpha}(t_n)$ is bounded from above for all $\alpha \in \mathcal{A}'$. Therefore, the right-hand side of [\(D.7\)](#) vanishes as $t_n \rightarrow \infty$, implying in turn that $v_{\alpha}(\bar{x}^*) = \bar{u}^*$ for all $\alpha \in \mathcal{A}'$, as claimed.

Now, since \bar{x}^* is a restricted equilibrium of \mathcal{G} , [Proposition 5.3](#) implies that it is a boundary rest point of [\(HD\)](#), so $\bar{u}^* = \langle v(\bar{x}^*) | \bar{x}^* \rangle < \langle v(\bar{x}^*) | p \rangle$ by [\(6.14\)](#). Moreover, since $\Psi(x) = \langle v(x) | p - x \rangle$, setting $t = t_n$ in [\(D.6\)](#) yields

$$t_n^{-1} \int_0^{t_n} \Psi(x(s)) ds = t_n^{-1} \int_0^{t_n} \langle v(x(s)) | p - x(s) \rangle ds = \langle v(\bar{x}(t_n)) | p \rangle - \bar{u}(t_n) \quad (\text{D.8})$$

so $\lim_{n \rightarrow \infty} t_n^{-1} \int_0^{t_n} \Psi(x(s)) ds = \langle v(\bar{x}^*) | p \rangle - \bar{u}^* > 0$. This establishes [\(D.6\)](#) for large enough $t = t_n$, completing our proof. \square

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