GAUSS'S LAW AND RESIDUE CALCULUS
IN THE FRAMEWORK OF COHOMOLOGY THEORY

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Abstract. In this paper, our aim is to incorporate Gauss's famous law in electrodynamics and some of Cauchy's results in residue calculus within the general setting of deRham cohomology, that will be shown to provide a most natural environment for their development. Namely, it will be demonstrated that forms defined and closed on punctured spaces can be assigned a residue similar to the one defined in complex analysis. This can be achieved in any number of dimensions and is, in essence, an extension of Gauss's three dimensional law: monopole fields generate the deRham cohomology groups of those spaces and with their help we will outline a method for evaluating certain kinds of multiple improper integrals.

1. Introduction

Gauss's Law in classical electrodynamics is an extremely elegant and deep result relating the flow of an electric field out of a given closed surface to the total electric charge within the surface. In a similar fashion, Cauchy's residue theorems connect the integral of a meromorphic function over a closed circuit with the function's residuesat the poles contained within the circuit. In this paper, we will try to combine these two fundamental results and extend them to spaces of higher dimension, with the help of cohomology theory which provides beautiful insights into the matter.

The purpose of this introductory section is two-fold: firstly, in an effort to be self-contained, we will provide clear definitions for most of the notions - especially the topological ones - that are to be used throughout this paper. Of course, we will not be original here, but we do urge the reader to be cautious before skipping the first part of this section as a number of concepts introduced *are* new. Finally, towards the end of the section, we will perform a few calculations, necessary for the main body of the paper to proceed smoothly.

To begin with, let A be a subset of $\mathbb{R}^n$ and consider $k \in \mathbb{N}$ with $k \leq n$. Recall that a *proper k-cube*[1] of A is a smooth C^∞ diffeomorphism $\gamma : I^k \mapsto A$ where I^k is just the k-dimensional unit cube: $I^k = [0,1]^k = [0,1] \times \cdots \times [0,1]$. When we use the term proper k-cube, we will refer either to the diffeomorphism γ itself or its image in A, depending on the context; however, when a distinction between the two must be made, we will denote the *image of γ in A* by $\langle\gamma\rangle$. Now, it is clear that a proper k-cube can be oriented in precisely two different ways [2], unless $k = 0$ - in which

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[1] As opposed to a *singular* k-cube - see [1] - that may possess self-intersections.

[2] Indeed, any proper k-cube is a compact topological space and, hence, admits a triangulation. Therefore, since a proper k-cube retains the homotopy type of a k-dimensional cube - see [2] - it is clear

case it admits but a single orientation. A proper k-cube, along with a choice of orientation will be simply called an *oriented k-cell*. Finally, if σ and τ are oriented k-cells that refer to the same proper k-cube taken with opposite orientations, we will - formally - write: $\sigma = -\tau$.

Now, if $\mathcal{S}_A$ is the collection of all oriented k-cells of A and $\mathbb{G}$ is a group, we may proceed to define a k-chain of A with coefficients in $\mathbb{G}$ as follows:

Definition 1.1. A *k-chain* of A with coefficients in $\mathbb{G}$ is a mapping $f : \mathcal{S}_A \mapsto \mathbb{G}$ which vanishes almost everywhere[3] on $\mathcal{S}_A$ and which satisfies $f(-\gamma) = -f(\gamma)$ for any oriented k-cell γ.

In the rest of this paper we will be concerned - almost exclusively - with k-chains whose coefficients are integers; if this is not the case, it will be stated explicitly. Also, in order to avoid cumbersome notation, if γ is an oriented k-cell, we will denote by γ the k-chain ϕ for which $\phi(\pm\gamma) = \pm 1$ and $\phi(\xi) = 0$ for any oriented k-cell ξ other than $\pm\gamma$. One can simply define addition of k-chains through addition of the respective mappings and it can easily be seen that - under addition - the set of k-chains of A forms a group, denoted by $C_k(A)$ and referred to as the k^{th} *chain group of A*. The k^{th} chain group of A can be thought of as a free group generated by the oriented k-cells of A and subject to the relations: $\gamma + (-\gamma) = 0$ for all $\gamma \in \mathcal{S}_A$.

A very important homomorphism that maps $C_k(A) \mapsto C_{k-1}(A)$ is the boundary homomorphism $\partial : C_k(A) \mapsto C_{k-1}(A)$ that assigns to an arbitrary k-chain C a (k-1)-chain ∂C that is made up of the "faces" of C [4]. We do not wish to provide a formal definition of ∂, as such a discussion would take us too far afield; as far as intuition goes, we will say that ∂ maps an oriented k-cell to its faces taken with the induced orientation and to see how ∂ acts on an arbitrary k-chain, one simply has to extend linearly - recall that the oriented k-cells are the generators of $C_k(A)$. Now, as can be seen in [1] and [2] the kernel of ∂ is a subgroup of $C_k(A)$ denoted by $Z_k(A) = \ker\partial = \{C \in C_k(A) : \partial C = 0\}$ and referred to as the k^{th} *cycle group of A*; if Z is an element of $Z_k(A)$ then Z will be called a *k-cycle of A*. In a similar fashion, the image of ∂ is a subgroup of $C_{k-1}(A)$ denoted by $B_{k-1}(A) = \mathrm{Im}\partial = \{C \in C_{k-1}(A) : \exists F \in C_k(A) \text{ with } \partial F = C\}$ and referred to as the $(k-1)^{\text{th}}$ *group of boundaries*; if $B \in B_{k-1}(A)$, then B will be called a *(k-1)-boundary* or a *bounding (k-1)-cycle*. Finally, we define the k^{th} *homology group of A* to be the quotient group $H_k(A) = Z_k(A)/B_k(A)$ whose elements are the equivalence classes of *homologous* k-cycles [5].

Let us consider a k-cycle Z of $\mathbb{R}^{k+1}$. Since $H_k(\mathbb{R}^{k+1})$ is trivial [6] we can see that Z must also be a k-boundary. In other words, there exists a (k+1)-chain F of $\mathbb{R}^{k+1}$ such that: $Z = \partial F$. In addition to that, F is unique; for, if G is another (k+1)-chain of $\mathbb{R}^{k+1}$ with $\partial G = Z$, it follows that $\partial G = \partial F$ or, equivalently: $\partial(G - F) = 0$ which shows $G - F$ to be a (k+1)-cycle. However, $Z_{k+1}(\mathbb{R}^{k+1})$ is also trivial, implying that

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that we can choose a triangulation that maps any proper k-cube to a k-simplex, and it is a well known fact that a k-simplex can be oriented in precisely two different ways - except, of course, when $k = 0$.

[3] That is, everywhere on S_A with the possible exception of a finite number of proper k-cubes.

[4] A formal definition of the boundary operator and of the faces of a k-chain can again be found in [1]. Also, [2] provides great intuition into the the matter.

[5] That is, k-cycles whose difference is a k-boundary.

[6] Again, see [2] or practically any other book on algebraic topology.

there are no (k+1)-cycles other than 0, the trivial one; hence, $G - F = 0$ or $G = F$ and we have shown that F is unique. Thus, we have arrived at the following definition:

Definition 1.2. Let C be a k-cycle of $\mathbb{R}^{k+1}$. The *unique* (k+1)-chain of $\mathbb{R}^{k+1}$ whose boundary is C will be called the *interior of* C and denoted by $\mathrm{Int}(C)$. More succinctly: $F = \mathrm{Int}(C) \Leftrightarrow \partial F = C$.

Now, let us consider the punctured space: $T = \mathbb{R}^{k+1} \setminus \{P\}$ where P is an arbitrary point of $\mathbb{R}^{k+1}$. Clearly, this space is homotopy equivalent to the k-sphere: $T \simeq S^k$. Hence, the k^{th} homology group of T will be: $H_k(T) \simeq H_k(S^k) \simeq \mathbb{Z}$ - see [2] for a complete discussion on the matter. Moreover, we can give the unit k-sphere centered at P the outward orientation [7] to obtain a k-cycle S. Since the homology class of S: $\Sigma = [S]$ is non-trivial, we can choose Σ to be the generator for the infinite cyclic group: $H_k(T) = \{\Sigma | \varnothing\}$. Therefore, if $Z \in Z_k(T)$ is a k-cycle of T, it follows that the homology class $[Z]$ [8] can be expressed in terms of the generator Σ as: $[Z] = n\Sigma$ where $n \in \mathbb{Z}$ - recall that $H_k(T) \simeq \mathbb{Z}$. This expression is obviously unique and leads us to:

Definition 1.3. Let Z be a k-cycle of $\mathbb{R}^{k+1} \setminus \{P\}$. The *unique* integer $n \in \mathbb{Z}$ for which: $[Z] = n\Sigma$ with Σ defined as above, will be referred to as the *index* or *winding number of Z with respect to the point P* and we will write:

$$n = \mathrm{Ind}(Z : O) \tag{1.1}$$

The notions of a k-cycle, its interior and its index with respect to a given point $P \in \mathbb{R}^{k+1}$ will play a principal part in this paper, since integration over k-cycles is what the form residues help us with. However, for the time being, we will leave topology aside and turn to the study of arrays which are also of great importance. So, let A be a set and consider $k_1, \dots k_m \in \mathbb{N}$:

Definition 1.4. A collection T of $k_1 \times \cdots \times k_m$ elements of A will be called an *array of type* $(k_1, \dots k_m)$ *with elements in A*. The *order* of T will be the number m while k_j will be referred to as the j^{th} *dimension* of T. The notation used is $T_{j_1 \dots j_n}$ for the element of A corresponding to $(j_1, \dots j_m)$ while the array T itself will be denoted by: $T = \left[T_{j_1 \dots j_n}\right]$.

A point of the above definition that deserves special mention, is the case of order zero: $m = 0$. In that case, the array T will consist of a single element $q \in A$ and we will write: $T = [q]$. Furthermore, in the rest of this paper, we will be using arrays of a more special form, namely those whose elements are real numbers and for which: $k_1 = \ldots = k_m = k$. Thus, henceforward and unless otherwise specified, an array T of type $\underbrace{(k, \dots, k)}_{\text{m times}}$ with elements in $\mathbb{R}$ will be simply called a - real - *array of order m and dimension k*.

As we shall see, one property of square matrices - that is, arrays of order 2 - that needs to be properly generalized in the case of a real array of order m, is the trace of the matrix. The exact motivation behind our definition cannot be presented here but it should become clear later in the paper when we examine whether the

[7] Actually, the term "outward" is well-defined only for $k \geq 2$. When $k = 1$, we will be giving S^1 the standard - anticlockwise - orientation.

[8] That is, the equivalence class of k-cycles of T that are homologous to Z

fundamental forms Ω_T are harmonic functions or not. Wihtout further ado, we have:

Definition 1.5. Let T be a real array of order $m \geq 2$ and dimension k. The *essence* of T is defined to be the array of order $m-2$ and dimension k:

$$\mathrm{ess}(T) = \left[\sum_{i=1}^{k} \sum_{\ell=1}^{m} \sum_{n=1}^{m} T_{i_1 \ldots i_{\ell-1} \alpha \ldots i_{n-1} \beta \ldots i_{m-2}} \delta_i^\alpha \delta_i^\beta \right]. \tag{1.2}$$

If T is of order $m = 0, 1$ or if $\mathrm{ess}(T)$ is a zero array, T will be called *neutral*.

Note : in the above definition, we are making use of the summation convention. To be more precise, a quantity such as: $T_{\ldots j \ldots} R^{\cdots j \cdots}$ is actually shorthand for $\sum_{j=1}^{k} T_{\ldots j \ldots} R^{\cdots j \cdots}$. Therefore, in relation (1.2), the indices α and β are summed over. Also, if the above definition is worked out in the case $m = 2$, it is easy to see that we will end up with two times the regular trace of the matrix, thus showing that up to an unimportant multiplicative constant, our definition is a valid extension of the trace of a matrix to higher order arrays.

The neutral arrays are exactly those arrays which lead to the "useful" singular terms, that are to be identified with the ordinary multipole fields encountered in physics. Indeed, we will prove that a singular term is a closed form if and only if the corresponding array is neutral, and this is one of the main points in this paper. However, in order to arrive to the definition of fundamental forms and singular terms, we will make heavy use of spherical symmetry in k-dimensional spaces; therefore, it is necessary to put aside arrays and introduce a system of "polar" coordinates in $\mathbb{R}^k$. So, let us consider the space of real numbers $\mathbb{R}^k$ and a system of Euclidean coordinates $(x_1, \ldots x_k)$ based at the point $O \in \mathbb{R}^k$ [9].

Definition 1.6. If $P(x_1, \ldots x_k)$ is a point of $\mathbb{R}^k \setminus \{O\}$ and $\rho_j = \sqrt{x_1^2 + \cdots + x_j^2}$, the *system of polar coordinates on* $\mathbb{R}^k$ *based at* O is defined by the following relations:

$$\begin{aligned} r &= \rho_k = \sqrt{x_1^2 + \cdots + x_k^2} \\ \omega_{k-1} &= \cos^{-1}\left(\frac{x_k}{r}\right) = \cos^{-1}\left(\frac{x_k}{\rho_k}\right) \\ &\vdots \\ \omega_j &= \cos^{-1}\left(\frac{x_{j+1}}{\sqrt{r^2 - x_k^2 - \cdots - x_{j+2}^2}}\right) = \cos^{-1}\left(\frac{x_{j+1}}{\rho_{j+1}}\right) \\ &\vdots \\ \omega_1 &= \cos^{-1}\left(\frac{x_2}{\sqrt{x_1^2 + x_2^2}}\right) = \cos^{-1}\left(\frac{x_2}{\rho_2}\right) \end{aligned} \tag{1.3}$$

In the above expressions, $\omega_j \in [0, \pi]$ for $1 < j \leq k-1$. However, when $j = 1$, we let $\omega_j \in [0, 2\pi]$ and ask in addition that $\sin \omega_1 = \frac{x_1}{\sqrt{x_1^2 + x_2^2}}$; in this way, all the

[9] From now on, and unless otherwise specified, we will assume that $k \geq 3$. Things are not fundamentally different for the case $k = 2$ and it will be addressed at the end of the following chapter.

ω_j's are well defined and we have a smooth bijection of Euclidean coordinates $(x_1, \dots x_k)$ to polar coordinates $(r, \omega_1, \dots \omega_{k-1})$. However, it should be emphasized that in the case $k = 3$ the above definition does not reduce to the usual spherical coordinates of $\mathbb{R}^k$: the angle ω_1 which should correspond to the azimuth ϕ is not measured with respect to the x_1 axis - we should have used sin in place of cos to agree with tradition. The definition is presented in this way, so that notation may be simplified later on when the forms ξ_i are introduced.

In order to gain intuition, one can express Euclidean coordinates x_i in terms of polar ones simply by inverting the expressions (1.3). By doing just that we obtain:

$$\begin{aligned} x_k &= r\cos\omega_{k-1} \\ &\vdots \\ x_j &= r\cos\omega_{j-1}\sin\omega_j\cdots\sin\omega_{k-1} \\ &\vdots \\ x_1 &= r\sin\omega_1\sin\omega_2\cdots\sin\omega_{k-1} \end{aligned} \tag{1.4}$$

Having thus defined the system of polar coordinates on $\mathbb{R}^k \setminus \{O\}$ we may ask what the Hodge star operator - $\star$ - will yield when applied to the forms $\mathrm{d}r, \mathrm{d}\omega_1 \dots \mathrm{d}\omega_{k-1}$. To do this, we must supply $\mathbb{R}^k$ with a nondegenerate scalar product $(\ ,\ )$ which we choose to be the standard Euclidean one: $(a, b) = \sum_{j=1}^{k} a_j b_j$. Under this product, one can verify the truth of the following lemma:

Lemma 1.7. *Consider the basis $\overline{\xi}$ of $\Omega^1(\mathbb{R}^k)$ defined by:*

$$\begin{aligned} \xi_1 &= r\sin\omega_2\cdots\sin\omega_{k-1}\,\mathrm{d}\omega_1 \\ &\vdots \\ \xi_j &= r\sin\omega_{j+1}\cdots\sin\omega_{k-1}\,\mathrm{d}\omega_j \\ &\vdots \\ \xi_{k-1} &= r\,\mathrm{d}\omega_{k-1} \\ \xi_k &= \mathrm{d}r \end{aligned} \tag{1.5}$$

Then, the basis $\overline{\xi}$ is orthonormal, that is: $(\xi_i, \xi_j) = \delta_{ij}$ and preserves orientation with respect to the standard basis $\overline{e} = \{\mathrm{d}x_1, \dots, \mathrm{d}x_k\}$.

Proof. To begin with, let us apply the d operator to relations (1.3). For the radial coordinate $r = \rho_k$ one obtains the usual expression:

$$\mathrm{d}r = \mathrm{d}\rho_k = \frac{x_1}{r}\mathrm{d}x_1 + \cdots + \frac{x_k}{r}\mathrm{d}x_k$$

with the angular expressions being:

$$\sin\omega_j\,\mathrm{d}\omega_j = \frac{x_{j+1}}{\rho_{j+1}^3}\Big(x_1\mathrm{d}x_1 + \cdots + x_j\mathrm{d}x_j\Big) - \frac{\rho_j^2}{\rho_{j+1}^3}\mathrm{d}x_{j+1}.$$

Therefore, if $1 \le j \le k-1$, we have:

$$\xi_j = \frac{x_{j+1}}{\rho_j\rho_{j+1}}\Big(x_1\mathrm{d}x_1 + \cdots + x_j\mathrm{d}x_j\Big) - \frac{\rho_j}{\rho_{j+1}}\mathrm{d}x_{j+1} \tag{1.6}$$

since it can be easily verified that: $\sin\omega_j = \frac{\rho_j}{\rho_{j+1}}$ and, as a direct consequence: $\sin\omega_{j+1}\cdots\sin\omega_{k-1} = \frac{\rho_{j+1}}{\rho_k}$. Finally, for the case $j = k$, one has:

$$\xi_k = \mathrm{d}r = \frac{x_1}{r}\mathrm{d}x_1 + \cdots + \frac{x_k}{r}\mathrm{d}x_k$$

and we have expressed all elements of $\overline{\xi}$ in terms of the standard basis elements $\mathrm{d}x_i$: $\xi_i = \sum_{j=1}^{k} B_{ij}\mathrm{d}x_j$ where $\mathcal{B} = [B_{ij}]$ denotes the change of basis matrix.

Now, proving the orthonormality of the basis $\overline{\xi}$ is only a straightforward calculation away: simply consider the products (ξ_i, ξ_j) and use relations (1.6) along with orthonormality of the basis $\overline{e}$ to get the desired result: $(\xi_i, \xi_j) = \delta_{ij}$.

In order to show that $\overline{\xi}$ preserves orientation with respect to $\overline{e}$ one need only show that $\det(\mathcal{B}) = 1 > 0$ and this can be achieved by performing a simple induction trick. Indeed, let $\mathfrak{D}_k$ denote the determinant:

$$\mathfrak{D}_k = \begin{vmatrix} \frac{x_2x_1}{\rho_2\rho_1} & -\frac{\rho_1}{\rho_2} & 0 & \dots & 0 \\ \frac{x_3x_1}{\rho_3\rho_2} & \frac{x_3x_2}{\rho_3\rho_2} & -\frac{\rho_2}{\rho_3} & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ \frac{x_kx_1}{\rho_{k-1}\rho_k} & \frac{x_kx_2}{\rho_{k-1}\rho_k} & \frac{x_kx_3}{\rho_{k-1}\rho_k} & \dots & -\frac{\rho_{k-1}}{\rho_k} \\ \frac{x_1}{\rho_k} & \frac{x_2}{\rho_k} & \frac{x_3}{\rho_k} & \dots & \frac{x_k}{\rho_k} \end{vmatrix} \tag{1.7}$$

It should be clear that $\mathfrak{D}_k$ is just $\det(\mathcal{B})$ - recall that locally $\mathcal{B}$ maps $\Lambda^1(\mathbb{R}^k) \mapsto \Lambda^1(\mathbb{R}^k)$. Moreover, if we expand $\mathfrak{D}_{k+1}$ in minors w.r.t. the $(k+1)^{\text{th}}$ column, we obtain:

$$\mathfrak{D}_{k+1} = -\frac{\rho_k}{\rho_{k+1}}\Big(-\frac{\rho_k}{\rho_{k+1}}\Big)\mathfrak{D}_k + \frac{x_{k+1}}{\rho_{k+1}}\frac{x_{k+1}}{\rho_{k+1}}\mathfrak{D}_k = \mathfrak{D}_k$$

since ρ_{k+1}^2 is just $\rho_k^2 + x_{k+1}^2$. Therefore, taking into account that $\mathfrak{D}_2 = 1$ as can be seen by an easy direct computation, the induction is complete and $\det(\mathcal{B}) = 1 > 0$ for any number of dimensions. This shows that $\overline{\xi}$ preserves orientation w.r.t. $\overline{e}$ and completes our proof. □

Lemma 1.7 can also be rephrased in terms of the change of basis transformation $\mathcal{B}$ by stating that $\mathcal{B}$ is an orientation preserving isometry. Therefore, since $\mathcal{B}$ leaves the volume form $\mu = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_k$ intact - that is: $\mu = \xi_1 \wedge \ldots \wedge \xi_k$ as well - we can easily deduce how the Hodge star operator will transform the forms ξ_j. Indeed, from the definition of $\star$ one has:

$$\star\xi_j \wedge \xi_j = (\xi_j, \xi_j)\mu = \xi_1 \wedge \ldots \wedge \xi_k = (-1)^{k-j}(\mu \setminus \xi_j) \wedge \xi_j$$

where $\mu \setminus \xi_j$ is shorthand for the form $\xi_1 \wedge \ldots \wedge \xi_{j-1} \wedge \xi_{j+1} \wedge \ldots \wedge \xi_k$. Therefore, since the $\star$ operator acting on 1-forms is an isomorphism - and, hence, an injection - mapping $\Lambda^1(\mathbb{R}^k)$ to $\Lambda^{k-1}(\mathbb{R}^k)$, we have proven the following lemma:

Lemma 1.8. *For the members of the basis $\overline{\xi}$ under the usual Euclidean product $(\ ,\ )$ one has:*

$$\star\xi_j = (-1)^{k-j}\mu \setminus \xi_j \tag{1.8}$$

where $\mu \setminus \xi_j$ stands for μ with ξ_j deleted, that is: $\mu \setminus \xi_j = \xi_1 \wedge \ldots \wedge \xi_{j-1} \wedge \xi_{j+1} \wedge \ldots \wedge \xi_k$.

The two results presented above will be of great use later on, when we evaluate the integral of a singular term over the (k-1)-dimensional sphere.

2. Singular Terms and Form Residues

In the theory of classical electrodynamics, it is a well known fact that any electric field with point singularities can be decomposed as the sum of an electric field with no singularities and a series - finite or infinite - of multipole fields - see for example [5]. In mathematical terminology, this means that a 1-form which is defined and closed on 3-space except for a subset A of $\mathbb{R}^3$ which is not dense in the space can be decomposed into a 1-form which is defined and closed on the whole of $\mathbb{R}^3$ and a series of 1-forms which correspond to the multipole fields. This result is exceedingly similar to the Laurent decomposition of meromorphic functions in complex analysis and is readily adaptable to higher dimensional spaces once the multipole fields have been worked out. These "fields" will be referred to as *singular terms* of the corresponding order and for the purposes of this paper we may begin as follows:

Definition 2.1. Consider a real array $T = [T_{j_1 \cdots j_m}]$ of order m and dimension k and let Ω_T denote the 0-form of $\mathbb{R}^k \setminus \{O\}$ defined by:

$$\Omega_T = \frac{T_{j_1 \cdots j_m} x^{j_1} \cdots x^{j_m}}{r^{2m+k-2}} \tag{2.1}$$

The 0-form Ω_T will be referred to as the *fundamental form corresponding to the array* T.

Note : in the above definition, we are making use of the summation convention. To be more precise, a quantity such as: $T_{\ldots j \ldots} R^{\cdots j \cdots}$ is actually shorthand for $\sum_{j=1}^{k} T_{\ldots j \ldots} R^{\cdots j \cdots}$.

The fundamental forms are the first step towards introducing the singular terms in which a given closed (k-1)-form will be decomposed. Hopefully, the motivation behind our definition will be clear to the reader before the end of this section but, at this point, no justification can be given without a long and thorough discussion on the matter.

Definition 2.2. Consider a real array $T = [T_{j_1 \ldots j_m}]$ of order m and dimension k. The $(k-1)$-form defined on $\mathbb{R}^k \setminus \{O\}$ by:

$$P_T = - \star \, \mathrm{d}\Omega_T$$

will be referred to as the *singular term corresponding to the array* T. The *order* of P_T is defined equal to the order of the array T and P_T will be frequently referred to as a 2^m*-pole*.

Before proceeding any further, we must deduce whether the fundamental forms are harmonic functions on $\mathbb{R}^k \setminus \{O\}$ a question which is clearly equivalent to asking whether the singular terms are closed on $\mathbb{R}^k \setminus \{O\}$. To this end, we will prove the following proposition which shows that the answer depends solely on the essence of the term's corresponding array.

Proposition 2.3. *Let $T = [T_{j_1 \dots j_m}]$ be a real array of order m and dimension k. Then, P_T is closed on $\mathbb{R}^k \setminus \{O\}$ iff T is neutral, that is:* $\mathrm{d}P_T = 0$ *iff* $\mathrm{ess}(T) = 0$.

Proof. To begin with, let us set: $\varepsilon_i = \frac{\partial \Omega_T}{\partial x_i}$; then, the closedness condition: $\mathrm{d}\Omega_T = 0$ is equivalent to asking that: $\displaystyle\sum_{i=1}^{k} \frac{\partial \varepsilon_i}{\partial x_i} = 0$. Having said that, the proof is but a few direct calculations away, and we will merely present an outline to them.

Indeed, if we let $\mathcal{T} = T_{j_1 \dots j_m} x^{j_1} \cdots x^{j_m}$ and set: $\mathcal{R}_i = \frac{\partial \mathcal{T}}{\partial x_i}$, we can easily see that:

$$\varepsilon_i = -(2m + k - 2) \frac{\mathcal{T} x_i}{r^{2m+k}} + \frac{\mathcal{R}_i}{r^{2m+k-2}}$$

Therefore, a second differentiation w.r.t. x_i will yield:

$$\frac{\partial \varepsilon_i}{\partial x_i} = \frac{2m + k - 2}{r^{2m+k}} \Big((2m + k) \mathcal{T} \frac{x_i^2}{r^2} - 2\mathcal{R}_i x_i - \mathcal{T} \Big) + \frac{1}{r^{2m+k-2}} \frac{\partial \mathcal{R}_i}{\partial x_i}. \tag{2.2}$$

However, the very definition of $\mathcal{R}_i$ infers:

$$\mathcal{R}_i = \sum_{\ell=1}^{m} T_{j_1 \dots j_{\ell-1} \alpha \dots j_m} x^{j_1} \cdots x^{j_{\ell-1}} \delta_i^{\alpha} \cdots x^{j_m}$$

and, if we multiply by x_i and sum over i, we will obtain:

$$\sum_{i=1}^{k} \mathcal{R}_i x_i = \sum_{\ell=1}^{m} \mathcal{T} = m\mathcal{T}.$$

Moreover, $\displaystyle\sum_{i=1}^{k} x_i^2 = r^2$; thus, relation 2.2 will yield:

$$\sum_{i=1}^{k} \frac{\partial \varepsilon_i}{\partial x_i} = \frac{1}{r^{2m+k-2}} \sum_{i=1}^{k} \frac{\partial \mathcal{R}_i}{\partial x_i}. \tag{2.3}$$

Now, a direct calculation will reveal that:

$$\sum_{i=1}^{k} \frac{\partial \mathcal{R}_i}{\partial x_i} = \sum_{i=1}^{k} \sum_{\ell=1}^{m} \sum_{n=1}^{m} T_{j_1 \dots j_{\ell-1} \alpha \dots j_{n-1} \beta \dots j_m} x^{j_1} \cdots x^{j_{\ell-1}} \delta_i^{\alpha} \cdots x^{j_{n-1}} \delta_i^{\beta} \cdots x^{j_m}$$

and, for P_T to be closed, this expression must vanish for all the x^i's. Therefore, we must ask that the array

$$\sum_{i=1}^{k} \sum_{\ell=1}^{m} \sum_{n=1}^{m} T_{i_1 \dots i_{\ell-1} \alpha \dots i_{n-1} \beta \dots i_{m-2}} \delta_i^{\alpha} \delta_i^{\beta}$$

be equal to zero. But, since this array is just $\mathrm{ess}(T)$ - compare with definition 1.5 - and vanishes if and only if T is neutral, we can see that P_T *is closed if and only if* T *is neutral*. □

This result will allow us to show that when a singular term corresponding to a neutral array is integrated over a $(k-1)$-cycle of $\mathbb{R}^k \setminus \{O\}$, the result will be zero unless the array is of order 0. To be more precise, we will prove the following proposition which is the proper extension of Gauss's law to higher dimensional spaces:

Proposition 2.4 (Gauss's Law)**.** *Let Z be a $(k-1)$-cycle of $\mathbb{R}^k \setminus \{O\}$ and T a neutral array of dimension k and order m. Then:*

$$\int_Z P_T = \begin{cases} q(k-2)\mathcal{A}_{k-1}\mathrm{Ind}(Z:O), & \textit{if } m=0 \textit{ and } T=[q] \\ 0, & \textit{if } m \geq 1 \end{cases} \tag{2.4}$$

where $\mathcal{A}_{k-1} = 2\,\frac{\sqrt{\pi}^k}{\Gamma\left(\frac{k}{2}\right)}$ is the area spanned by the unit (k-1)-sphere.

Proof. Let $[Z]$ be the homology class of Z and let Σ denote the homology class of the unit (k-1)-sphere S^{k-1}, taken with the outward orientation and centered at O. From the definition of $\mathrm{Ind}(Z:O)$ we have:

$$[Z] = \mathrm{Ind}(Z:O)\Sigma \Rightarrow [Z - \mathrm{Ind}(Z:O)S^{k-1}] = 0$$

Obviously, this implies the existence of a k-chain C of $\mathbb{R}^k \setminus \{O\}$ with the property: $\partial C = Z - \mathrm{Ind}(Z:O)S^{k-1}$ [10]. Therefore, an application of Stokes' theorem - see for example [4], [1] or [6] - easily yields:

$$\int_Z P_T - \mathrm{Ind}(Z:O)\int_{S^{k-1}} P_T = \int_{\partial C} P_T = \int_C \mathrm{d}P_T = 0$$

(recall that P_T is closed on $\mathbb{R}^k \setminus \{O\}$ since the array T is neutral). Hence, we obtain:

$$\int_Z P_T = \mathrm{Ind}(Z:O)\int_{S^{k-1}} P_T \tag{2.5}$$

and it suffices to evaluate the integral of P_T over the (k-1) unit sphere S^{k-1}, centered at O and taken with the outward orientation.

Bearing this in mind, consider the sphere S_ε - which is the same as S^{k-1} except that it is of radius ε - and the set:

$$J_\varepsilon = \{\varepsilon\} \times [0, 2\pi] \times \underbrace{[0,\pi] \times \cdots \times [0,\pi]}_{\text{k}-2 \text{ times}}$$

One can readily see that J_ε is exactly where S_ε is pulled back to under ϕ - the transformation from polar to euclidean coordinates of $\mathbb{R}^k$ - and we are left to evaluate $\int_{\phi(J_\varepsilon)} P_T$. This task can be accomplished by switching to polar coordinates, where we get:

$$\int_{\phi(J_\varepsilon)} P_T = \int_{J_\varepsilon} \phi^* P_T = -\int_{J_\varepsilon} \phi^*(\star \mathrm{d}\Omega_T)$$

In order to proceed, we will have to express Ω_T in terms of the forms ξ_j introduced in section 1. Now, to simplify notation, consider the functions: $\alpha_i = \frac{x_i}{r}$ where x_i is expressed in polar coordinates via relations (1.4); as a direct consequence, one has: $\Omega_T = \frac{T_{j_1\dots j_m}\alpha^{j_1}\cdots\alpha^{j_m}}{r^{m+k-2}}$. Therefore, taking d of Ω_T, will yield:

$$\mathrm{d}\Omega_T = -(m+k-2)\frac{T_{j_1\dots j_m}\alpha^{j_1}\cdots\alpha^{j_m}}{r^{m+k-1}}\mathrm{d}r \;+\; \text{terms containing } \mathrm{d}\omega_i.$$

The differential above can be expressed in terms of the basis $\overline{\xi}$ simply by considering that ξ_j is a multiple of $\mathrm{d}\omega_j$. Thus, by applying the star operator to $\mathrm{d}\Omega_T$, the terms

[10] It is not hard to show that this k-chain is, in fact, unique and equal to $\mathrm{Int}(Z) - \mathrm{Ind}(Z:O)B_k$ where B_k is the ball bounded by S^{k-1}.

containing $\mathrm{d}\omega_i$ will become wedge products containing the form $\xi_k = \mathrm{d}r$ and these terms will clearly vanish when pulled back by ϕ and integrated over J_ε - where r is kept fixed to ε. As a consequence of the above, we may write:

$$\begin{aligned}\int_{S_\varepsilon} P_T &= -\int_{J_\varepsilon} \phi^*(\star \mathrm{d}\Omega_T) = -\int_{J_\varepsilon} \frac{\partial \Omega_T}{\partial r} \xi_1 \wedge \ldots \wedge \xi_{k-1} \\ &= (m+k-2) \int_{J_\varepsilon} \frac{T_{j_1 \ldots j_m} \alpha^{j_1} \cdots \alpha^{j_m}}{r^{m+k-1}} r^{k-1} \tau\end{aligned}$$

where τ is defined by:

$$\tau = \frac{1}{r^{k-1}} \xi_1 \wedge \cdots \wedge \xi_{k-1} = \sin \omega_2 \ldots \sin^{k-2} \omega_{k-1} \, \mathrm{d}\omega_1 \wedge \ldots \wedge \mathrm{d}\omega_{k-1}$$

and corresponds to the "solid angle" form of $\mathbb{R}^k$. So, if we let

$$J = [0, 2\pi] \times \underbrace{[0, \pi] \times \cdots \times [0, \pi]}_{\text{k}-2 \text{ times}}$$

and set $r = \varepsilon$ in the above integrals, we will obtain the crucial result:

$$\int_{S_\varepsilon} P_T = \frac{m+k-2}{\varepsilon^m} \int_J T_{j_1 \ldots j_m} \alpha^{j_1} \cdots \alpha^{j_m} \tau \tag{2.6}$$

For $m = 0$ and $T = [q]$, the above result reduces to: $\int_{S_\varepsilon} P_T = q(k-2) \int_J \tau$. However, $\int_J \tau$ is just the solid angle bounded by the sphere S^{k-1} with respect to O; in other words $\int_J \tau = \mathcal{A}_{k-1}$ where $\mathcal{A}_{k-1} = 2\frac{(\sqrt{\pi})^k}{\Gamma\left(\frac{k}{2}\right)}$ is the area spanned by the unit sphere S^{k-1}. Therefore, in conjunction with relation (2.5), we finally get[11]:

$$\int_Z P_T = q(k-2)\mathcal{A}_{k-1} \mathrm{Ind}(Z : O)$$

On the other hand, if $m \geq 1$, consider $\delta \neq \varepsilon$; then, by applying (2.6) we get

$$\int_{S_\delta} P_T = \frac{m+k-2}{\delta^m} \mathcal{I}_T$$

where $\mathcal{I}_T = \int_J T_{j_1 \ldots j_m} \alpha^{j_1} \cdots \alpha^{j_m} \tau$. But, since P_T is closed on $\mathbb{R}^k \setminus \{O\}$, we can use Stokes' theorem to show that: $\int_{S_\varepsilon} P_T = \int_{S_\delta} P_T$ which, combined with our previous results, leads to:

$$\frac{m+k-2}{\delta^m} \mathcal{I}_T = \frac{m+k-2}{\varepsilon^m} \mathcal{I}_T$$

Of course, this cannot hold for $m \geq 1$, unless $\mathcal{I}_T = 0$. This shows that $\displaystyle\int_Z P_T$ must vanish for $m \geq 1$ and completes our proof. $\square$

[11] In order to obtain the integral over the unit sphere S^{k-1}, it suffices to set $\varepsilon = 1$ in (2.6).

Along with definitions 2.1 and 2.2, the two results presented above clearly demonstrate how fundamental forms correspond to the potential of an ideal multipole in k-dimensional spaces, with singular terms corresponding to the fields themselves. Indeed, if we ignore the presence of the Hodge star operator which has been included for purposes of integration, we can see that the vector fields on $\mathbb{R}^k \setminus O$ associated to the singular terms have a single point singularity, both zero curl and zero divergence and, most important, they also respect Gauss's Law. Hence, they are proper extensions of multipoles to $\mathbb{R}^k$ with $k \geq 3$ and the reader is actively encouraged to treat them as such; henceforward, intuitive discussion based on the matter will be based freely upon this fact.

Now, before proceeding any further, we will have to present a few elements of deRham cohomology theory which provides a natural setting for the study of closed forms. Of course, we will not venture into detailed discussions as this would take us beyond the scope of this paper; we will only go as far as to define the k^{th} - deRham - cohomology space of a space A.

To be more precise, let us consider a submanifold A of $\mathbb{R}^n$ and a differential k-form $\omega \in \Omega^k(A)$. Under addition of k-forms and multiplication by a scalar $\lambda \in \mathbb{R}$, the set of closed k-forms: $Z^k(A) = \{\omega \in \Omega^k(A) : \mathrm{d}\omega = 0\}$ attains the structure of a *vector space* over the real field $\mathbb{R}$ and will be referred to as the k^{th} *cocycle space of A*. Moreover, if there exists a (k+1)-form $\sigma \in \Omega^{k+1}(A)$ such that: $\omega = \mathrm{d}\sigma$, we will say that ω is exact on A. The set of exact forms: $B^k(A) = \{\omega \in \Omega^k(A) : \exists\sigma \in \Omega^{k+1}(A) \text{with } \mathrm{d}\sigma = \omega\}$ also has the structure of a real vector space and will be referred to as the k^{th} *coboundary space of A*. Finally, if $\omega \in \Omega^k(A)$, we define the - deRham - *cohomology class* of ω to be the set: $[\omega] = \{\psi \in Z^k(A) : \omega - \psi \in B^k(A)\}$. Bearing this in mind, the k^{th} - *deRham - cohomology space of A* is defined to be the quotient space: $H^k(A) = Z^k(A)/B^k(A) = \dfrac{\ker \mathrm{d}}{\operatorname{Im} \mathrm{d}} = \{[\omega] : \omega \in \Omega^k(A)\}$ - see [6] and [3].

A very deep and greatly celebrated theorem by deRham - see [6] - states that for a given smooth manifold M, the spaces $H^k(M)$ and $H_k(M)$ are isomorphic[12], the isomorphism being given by integration. Therefore, if A is a punctured open ball of of $\mathbb{R}^{k+1}$ with the interior point Q removed, we will have: $H^k(A) \simeq H_k(A) \simeq \mathbb{R}$. Furthermore, from proposition 2.4 we may deduce that the singular term P corresponding to the array of order 0: $T = [1]$ and based at Q is a closed k-form which is not exact[13]. Therefore, since P is not exact, the cohomology class $\pi = [P]$ of P will be non-trivial: $\pi \neq 0$ and, consequently, we may choose π as a basis for $H^k(A)$. So, if $\omega \in Z^k(A)$ is a closed k-form of A, we may write: $[\omega] = q \cdot \pi$ where q is a *unique* real number, and we have arrived at the following definition:

Definition 2.5. Let ω be a closed k-form on a punctured open (k+1)-ball A with an interior point Q removed. The - unique - $q \in \mathbb{R}$ for which: $[\omega] = q \cdot \pi$ with π defined as above, will be called the *residue of ω at Q* and we will write:

$$\operatorname{Res}(\omega : Q) = q \tag{2.7}$$

[12] In this paper, we have not given the structure of a vector space to the homology *groups* $H_k(M)$. However, the homology groups can easily attain this structure if we agree to let k-chains take on real coefficients - see [2] and [6] for a first discussion on the matter.

[13] Indeed, if it were exact, its integral over any k-cycle Z would have to be zero, which clearly does not hold if $\operatorname{Ind}(Z : Q) \neq 0$.

Note: There is a striking resemblance between the *residue* of a given closed form at a point Q and the *index* of a k-cycle with respect to the same point. In a sense, these notions are *dual* to one another, much as the notion of a k-form is dual to that of a k-chain. With this in mind, as the k-spheres centered at the "holes" of a manifold generate the k^{th} homology group of the manifold, the monopoles based at the same "holes" provide a basis for the k^{th} - deRham - cohomology space of the manifold. We will have more to say on this subject later on in this chapter.

One final point that we should stress, is that definition 2.5 allows us to define the residue of a closed k-form ω on A, even when A is an open (k+1)-ball with a - finite - collection of points $\mathcal{Q} = \{Q_j, j = 1 \dots n\}$ removed. To obtain the residue of ω at a given point Q_j one simply has to consider a neighbourhood of Q_j which contains no other "singular" points[14] and apply definition 2.5[15].

We will see that definition 2.5 is indeed a proper generalization of the traditional Cauchy residue found in Complex Analysis. To be more precise, we will be able to derive the following proposition from our previous discussion:

Proposition 2.6. *Let A be an open ball of $\mathbb{R}^k$ with a point Q removed and consider a* closed *(k-1)-form $\omega \in \Omega^{k-1}(A)$. Then, if Z is a (k-1)-cycle of A, the integral of ω over Z is:*

$$\int_Z \omega = (k-2)\mathcal{A}_{k-1} \cdot \mathrm{Ind}(Z : Q) \cdot \mathrm{Res}(\omega : Q) \tag{2.8}$$

Proof. Let T denote the array of order 0: $T = [1]$. Then, from the definition of the residue of ω at Q, we have: $[\omega] = \mathrm{Res}(\omega : Q) \cdot \pi$ where $\pi = [P_T]$ and P_T is the - (k-1)-dimensional - singular term associated to T and based at Q. Therefore, there exists an exact (k-1)-form σ such that: $\omega = \sigma + \mathrm{Res}(\omega : Q)P_T$ and, by integrating over Z, we obtain:

$$\int_Z \omega = \int_Z \sigma + \mathrm{Res}(\omega : Q) \int_Z P_T.$$

But, $\int_Z \sigma$ clearly vanishes since σ is exact, while an application of proposition 2.4 will yield: $\displaystyle\int_Z P_T = (k-2)\mathcal{A}_{k-1}\mathrm{Ind}(Z : Q)$. Thus, by combining these results, we finally get:

$$\int_Z \omega = (k-2)\mathcal{A}_{k-1} \cdot \mathrm{Ind}(Z : Q) \cdot \mathrm{Res}(\omega : Q),$$

as was to be shown. □

As a direct consequence of the above, we may also obtain the following corollary - which is, in essence, a more intuitive way of rephrasing proposition 2.6:

Corollary 2.7. *Let A be a punctured open ball of $\mathbb{R}^k$ with an interior point Q removed, and consider a closed (k-1)-form ω of A. If S is a (k-1)-sphere contained in A, centered at Q and given the standard - outward - orientation, then we have:*

$$\mathrm{Res}(\omega : Q) = \frac{1}{(k-2)\mathcal{A}_{k-1}} \int_S \omega \tag{2.9}$$

[14] Recall that this is possible because the subset of singularities of ω is *not* dense in A.

[15] Alternatively, one could argue that $H^k(A)$ is an n-dimensional vector space and choose the set of monopoles based at the points Q_j as basis vectors for $H^k(A)$.

Now, we are ready to state and prove the central result of this paper which can be thought of as a direct generalization to higher dimensional spaces of Cauchy's main residue theorem. Without further ado, we have:

Proposition 2.8 (Cauchy's Residue Theorem). *Consider an open ball A of $\mathbb{R}^k$ and a differential (k-1)-form ω which is defined and closed on A except for a nondense collection of - interior - points $\mathcal{Q} = \{Q_j \in A, j = 1 \dots n\}$, that is: $\omega \in Z^{k-1}(A \setminus \mathcal{Q})$. Then, if Z is a (k-1)-cycle of $A \setminus \mathcal{Q}$, the following formula holds:*

$$\int_Z \omega = (k-2)\mathcal{A}_{k-1} \sum_{j=1}^{n} \mathrm{Ind}(Z : Q_j)\mathrm{Res}(\omega : Q_j) \tag{2.10}$$

Proof. Consider the array of order 0: $T = [1]$ and denote the - (k-1)-dimensional - singular term associated to T and based at Q_j by P_j. Obviously, $H^{k-1}(A \setminus \mathcal{Q}) \simeq \mathbb{R}^n$ and this allows us to choose the cohomology classes: $\pi_j = [P_j]$ as basis vectors for $H^{k-1}(A \setminus \mathcal{Q})$. Therefore, since ω is closed on $A \setminus \mathcal{Q}$, we will have: $[\omega] = \sum_{j=1}^{n} \pi_j$ or, equivalently:

$$\omega = \sigma + \sum_{j=1}^{n} \mathrm{Res}(\omega : Q_j)P_j$$

where σ is an exact (k-1)-form defined on $A \setminus \mathcal{Q}$. Moreover, it can be readily seen that the homology class of Z is just: $[Z] = \sum_{i=1}^{n} \mathrm{Ind}(Z : Q_i)\Sigma_i$ where Σ_i is the homology class of a sphere S_i centered at Q_i, taken with the outward orientation and contained in A - but containing no other elements of $\mathcal{Q}$. Consequently, there exists a k-chain C of $A \setminus \mathcal{Q}$ such that:

$$Z = \partial C + \sum_{i=1}^{n} \mathrm{Ind}(Z : Q_i)S_i.$$

- compare with proposition 2.4 as well. It should be clear that since ω is closed on $A \setminus \mathcal{Q}$, the integral $\int_{\partial C} \omega$ will vanish; in addition to that, when σ is integrated over a sphere S_i, it will also yield zero since σ is exact. Bearing this in mind, we obtain:

$$\begin{aligned}
\int_Z \omega &= \int_{\partial C} \omega + \sum_{i=1}^{n} \mathrm{Ind}(Z : Q_i) \int_{S_i} \omega \\
&= \sum_{i=1}^{n} \mathrm{Ind}(Z : Q_i) \int_{S_i} \Big(\sigma + \sum_{j=1}^{n} \mathrm{Res}(\omega : Q_j)P_j\Big) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \mathrm{Ind}(Z : Q_i)\mathrm{Res}(\omega : Q_j) \int_{S_i} P_j
\end{aligned}$$

However, given that $\mathrm{Res}(P_j : Q_j) = 1$, if we apply propositions 2.4 and 2.6 in order to evaluate $\int_{S_i} P_j$ we will get:

$$\int_{S_i} P_j = (k-2)\mathcal{A}_{k-1}\mathrm{Ind}(S_i : Q_j) = (k-2)\mathcal{A}_{k-1}\delta_{ij}$$

as S_i contains no other "singular" points of $\mathcal{Q}$ except for Q_i. Thus, we obtain:

$$\int_Z \omega = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{Ind}(Z : Q_i)\mathrm{Res}(\omega : Q_j)(k-2)\mathcal{A}_{k-1}\delta_{ij} \Rightarrow$$

$$\int_Z \omega = (k-2)\mathcal{A}_{k-1}\sum_{j=1}^{n} \mathrm{Ind}(Z : Q_j)\mathrm{Res}(\omega : Q_j)$$

and our proof is complete. □

With the aid of this proposition, we may prove an extension to Poincaré's lemma[16] which will provide us with a criterion of whether a given closed form is exact or not. To be more precise, we will prove the following proposition:

Proposition 2.9. *Let M be a k-submanifold of $\mathbb{R}^k$ which is diffeomorphic to an open k-ball of $\mathbb{R}^k$, possibly with a - nondense - collection of points removed. Suppose further that $\omega \in \Omega^q(M)$ is a differential q-form of M which satisfies: $\int_Z \omega = 0$ for all q-cycles $Z \in Z_q(M)$. Then, ω is exact on M.*

Proof. First, we will attack the case $q = k-1$, which is of the most interest to us. So, let us assume that M is diffeomorphic to $A = B \setminus \mathcal{Q}$ where B is an open k-ball and $\mathcal{Q} = \left\{Q_j \in B, j = 1\dots n\right\}$ is a - finite - collection of points of B. Moreover, denote by ϕ the diffeomorphism that maps $A \mapsto M$: $M = \phi(A)$. Then, if W is a (k-1)-cycle of A, the image of W under ϕ will be a (k-1)-cycle $Z = \phi(W)$ of M and we may write:

$$\int_W \phi^*\omega = \int_{\phi(W)} \omega = \int_Z \omega = 0$$

a result which holds *for all* (k-1)-cycles of A.

Consequently, if we set $\psi = \phi^*\omega$ and let P_j denote the singular term of A which corresponds to the array $T = [1]$ and is based at Q_j, we will have:

$$\psi = \sigma + \sum_{j=1}^{n} \mathrm{Res}(\psi : Q_j)P_j$$

where σ is exact on A. However, if we integrate this expression over an arbitrary (k-1)-cycle W, we will obtain:

$$\int_W \psi = 0 \Rightarrow \sum_{j=1}^{n} \mathrm{Res}(\psi : Q_j)\mathrm{Ind}(W : Q_j) = 0$$

Clearly, this can hold *for all* (k-1)-cycles W of A if and only if $\mathrm{Res}(\psi : Q_j) = 0$ *for all* $j = 1\dots n$. Thus, $\psi = \sigma$ and we have shown that ψ is exact. Therefore, if $\tau \in \Omega^k(A)$ is a k-form of A with: $\mathrm{d}\tau = \psi$, by pushing forward w.r.t. ϕ, we obtain:

$$\mathrm{d}(\phi_*\tau) = \phi_*\mathrm{d}\tau = \phi_*\psi = \phi_*(\phi^*\omega) = \omega$$

and we have shown that ω *is exact on* M.

[16] Poincaré's lemma states that if ω is a closed form on a contractible k-manifold M, then ω has to be exact as well - see [6], [4], [1] and [3] for a discussion on the matter.

On the other hand, let $\omega \in \Omega^q(M)$ with $q < k-1$ and consider q-chains $A_1, A_2 \in C_q(M)$ which share the same boundary: $\partial A_1 = \partial A_2 = C \in C_{q-1}(M)$. It is not hard to show that:

$$\int_{A_1} \omega = \int_{A_2} \omega.$$

Indeed, if we set $Z = A_2 - A_1$, it should be clear that Z is a q-cycle: $\partial Z = \partial A_2 - \partial A_1 = C - C = 0$. However, by assumption, $\int_Z \omega = 0$, and this leads us to our previous result. With this in mind, we may proceed to define a linear function $\tilde{\sigma} : B_{q-1}(M) \mapsto \mathbb{R}$ which maps $C \in B_{q-1}(M)$[17] to $\tilde{\sigma}[C] = \displaystyle\int_F \omega$ where F is a q-chain satisfying: $\partial F = C$ - note that $\tilde{\sigma}$ is well defined since, if we choose another q-chain G with $\partial G = C$ we would have: $\int_G \omega = \int_F \omega = \tilde{\sigma}[C]$. Moreover, since $\tilde{\sigma}$ is defined via *integration* of q-forms, we can see that it is at least C^1 and we can use the smooth version of Tietze's extension theorem - see [6] and [2] - to extend $\tilde{\sigma}$ to a C^1 linear function $\bar{\sigma}$ on $C_{q-1}(M)$ which agrees with $\tilde{\sigma}$ on $B_{q-1}(M)$ - of course, this extension *is* not and *need* not be unique. Thus, if d denotes the coboundary operator, mapping $\mathrm{d} : C^{q-1}(M) \mapsto C^q(M)$[18] we know for a fact that the following diagram commutes - see for example [4]:

$$\begin{array}{ccc} C^{q-1}(M) & \xrightarrow{\mathrm{d}} & C^q(M) \\ \uparrow & & \uparrow \\ \Omega^{q-1}(M) & \xrightarrow{\mathrm{d}} & \Omega^q(M) \end{array}$$

Denote by $\bar{\omega}$ the linear function on $C^q(M)$ that corresponds to ω. Then, if F is an arbitrary q-chain with $\partial F = C$, we can apply the coboundary operator d to $\bar{\sigma}$ in order to obtain:

$$\mathrm{d}\bar{\sigma}[F] = \bar{\sigma}[\partial F] = \int_F \omega = \bar{\omega}[F].$$

This shows that $\bar{\omega} = \mathrm{d}\bar{\sigma}$. Therefore, if $\sigma \in \Omega^{q-1}(A)$ denotes the form that corresponds [19] to $\bar{\sigma} \in C^{q-1}(M)$, the commutativity of the diagram above shows that: $\mathrm{d}\sigma = \omega$ and ω is exact on M as was to be shown. □

Now, from the proof of this exactness criterion, we can readily obtain the following very useful corollary:

Corollary 2.10. *Let M be a k-submanifold of $\mathbb{R}^k$ which is diffeomorphic to an open k-ball of $\mathbb{R}^k$, possibly with a - finite - number of points removed. Then, a (k-1)-form ω will be exact on M iff:*

$$\mathrm{Res}(\phi^* \omega : Q_j) = 0$$

for all $Q_j \in \boldsymbol{Q}$ *- with ϕ and $\boldsymbol{Q}$ defined as above*[20].

In order to see how Poincaré's lemma can be derived from proposition 2.9, consider a contractible manifold M and a closed k-form ω on M: $\mathrm{d}\omega = 0$. Then, if

[17] Here, we are assuming that the (q-1)-chains take on real coefficients

[18] $C^q(M)$ refers to the space of q-cochains of M - see [4] for a full account.

[19] Again, we refer the reader to [4] for a clear account of how a q-cochain is assigned to a differential q-form.

[20] See proof of proposition 2.9

Z is an arbitrary k-cycle of M, Z will be a *bounding* k-cycle as well, i.e. there exists a (k+1)-chain C of M with: $\partial C = Z$. So, integrating ω over Z will yield:

$$\int_Z \omega = \int_{\partial C} \omega = \int_C \mathrm{d}\omega = 0$$

as ω is closed on M. Therefore, since this results holds for *any* k-cycle Z, we can conclude from proposition 2.9 that ω is exact on M, as was to be shown.

So, let us consider a k-form ω defined and closed on a deleted neighbourhood A of a point $Q \in \mathbb{R}^{k+1}$ and assume that it decomposes into singular terms as follows[21]:

$$\omega = \omega_0 + \sum_{j=0}^{m} P_{T_j}$$

with the singular terms P_{T_j} being based at Q and m - possibly equal to ∞ - the *order* of the pole at Q. From definition 2.5 we may deduce the following lemma:

Lemma 2.11. *If $T_0 = [q]$, then the* residue *of ω at Q is just:*

$$\mathrm{Res}(\omega : Q) = q. \tag{2.11}$$

In other words, the residue *of a closed form at a given point, is equal to the "coefficient" of the singular term of order 0, based at that point.*

Proof. By assumption, the singular decomposition of ω at P is:

$$\omega = \omega_0 + \sum_{j=0}^{m} P_{T_j},$$

where $T_0 = [q]$ and we are asked to show that $\mathrm{Res}(\omega : Q) = q$. Now, if Z is a k-cycle of A, its interior Int(Z) will be a k-chain of $A \cup Q$ and, integrating ω_0 over Z infers:

$$\int_Z \omega_0 = \int_{\mathrm{Int(Z)}} \mathrm{d}\omega_0 = 0$$

since ω_0 is closed on $A \cup Q$[22]. Furthermore, if $j \neq 0$, we can deduce from proposition 2.4 that: $\int_Z P_{T_j} = 0$ as well, and these results hold for *any* k-cycle of A. Therefore, proposition 2.9 tells us that both ω_0 and all of the P_{T_j} singular terms - with $j \neq 0$ - are exact on A, and we may write:

$$\omega = \sigma + P_{T_0}$$

where σ is an exact k-form of A. Thus, for the - deRham - cohomology class of ω, we obtain: $[\omega] = [P_{T_0}]$.

On the other hand, from definition 2.5 we know that: $[\omega] = \mathrm{Res}(\omega : Q)[P_Q]$ where P_Q is the singular term of A that corresponds to the array $T = [1]$ and which is based at Q. But, it can be easily seen that P_{T_0} is just $q \cdot P_Q$ and, by comparing the two expressions for $[\omega]$ we finally get: $\mathrm{Res}(\omega : Q) = q$. □

[21]Clearly, any such decomposition is unique.

[22]Recall that Z = ∂Int(Z).

One point in the above proof that we should stress is the exactness of singular terms of higher order. This result is of great intuitive value since it exposes the true nature of monopole terms, and deserves special mention, eventhough it is easily implied by previous results:

Corollary 2.12. *Let $Q \in \mathbb{R}^{k+1}$. Then,* **all** *singular terms based at Q are exact, except for the monopoles.*

This is the same kind of duality that we encountered earlier when dealing with the index of a k-cycle and the residue of a k-form. We know that all k-cycles of $\mathbb{R}^{k+1} \setminus O$ are also k-boundaries except for those that are homologous to the k-sphere centered at the origin. Similarly, all k-forms that are *closed* on $\mathbb{R}^{k+1} \setminus O$ are exact as well, except for those which are *co*homologous to the k-monopole based at the origin. The k-spheres generate the kth homology group of punctured spaces while the k-monopoles generate the corresponding cohomology space; with this kind of duality in mind, one can say that monopoles play the same part in function spaces as spheres play in ordinary geometrical ones.

Now, we are left to actually *evaluate* the residue of a k-form at one of its "singular" points. A priori, this need not be easy, as the form's expression in an arbitrary coordinate system might be quite complicated, and a singular decomposition would not be readily apparent. Therefore, we will present a *method* for computing the residue of a closed form that closely mimics the "shortcuts" used in Complex Analysis to compute the residue of an meromorphic function. The method is described in the following proposition:

Proposition 2.13. *Let A be an open ball of $\mathbb{R}^k$ and consider a (k-1)-form ω closed on A, with a pole of order m at a point $P \in A$. Then, the residue of ω at P is equal to:*

$$\operatorname{Res}(\omega : P) = \frac{1}{m!(k-2)} \lim_{r\to 0} \frac{\partial^m}{\partial r^m}\Big(r^{m+k-2}\vec{L}_\omega \cdot \vec{r}\Big) \tag{2.12}$$

where $\vec{L}_\omega = (\star^{-1}\omega)^\sharp$ and r is the usual polar coordinate, based at P[23]*.*

Proof. Consider a polar coordinate system $\{r, \omega_1, \dots, \omega_{k-1}\}$ based at P; since ω presents a pole of order m at P, it will decompose into singular terms as follows: $\omega = \omega_0 + \sum_{j=0}^m P_{T_j}$ where ω_0 is a (k-1)-form defined and closed on A. Also, set $\vec{L} = (\star^{-1}\omega_0)^\sharp$ and $\vec{L}_j = (\star^{-1}P_{T_j})^\sharp$ so that: $\vec{L}_\omega = \vec{L} + \sum_{j=0}^m \vec{L}_j$. Then, from the definition of singular terms, it follows that:

$$\vec{L}_j \cdot \vec{r} = -\Big(\mathrm{d}\Omega_{T_j}\Big)^\sharp \cdot \vec{r} = -\vec{\nabla}\Omega_{T_j} \cdot \vec{r} = -r\frac{\partial\Omega_{T_j}}{\partial r}.$$

Now, if we denote the array T_j by: $T_j = \Big[T^j_{s_1 \dots s_j}\Big]$ and, in addition, set $R_j = T^j_{s_1\dots s_j}\alpha^{s_1}\cdots\alpha^{s_j}$, we obtain:

$$\sum_{j=0}^m r^{m+k-2}\vec{L}_j \cdot \vec{r} = \sum_{j=0}^m r^{m-j}(j+k-2)R_j$$

[23]For the definition of the *index raising operator* $\sharp$ see [6]. Roughly speaking, if ω is a 1-form: $\omega = f_1\mathrm{d}x_1 + \cdots + f_k\mathrm{d}x_k$ then $\omega^\sharp = f_1\vec{e}_1 + \cdots + f_k\vec{e}_k$ with $\vec{e}_j$ being the standard vector basis for $\mathbf{R}^k$.

since R_j depends only on the angular coordinates ω_i and is independent of r. Now, by differentiating our previous result m times w.r.t. r, we get:

$$\frac{\partial^m}{\partial r^m}\sum_{j=0}^{m} r^{m+k-2}\vec{L}_j\cdot\vec{r}=\sum_{j=0}^{m} p_j r^{-j}(j+k-2)R_j$$

where $p_j=\prod_{\ell=0}^{m-1}(m-j-\ell)$. However, p_j vanishes for all j, unless $j=0$ in which case: $p_0=\prod_{\ell=0}^{m-1}(m-\ell)=m!$. Thus, we get:

$$\frac{\partial^m}{\partial r^m}\sum_{j=0}^{m} r^{m+k-2}\vec{L}_j\cdot\vec{r}=m!(k-2)R_0$$

But, R_0 is the sole element of the array T_0, and this is exactly the residue of ω at P. In other words, we have shown that:

$$\operatorname{Res}(\omega:P)=\frac{1}{m!(k-2)}\frac{\partial^m}{\partial r^m}\sum_{j=0}^{m} r^{m+k-2}\vec{L}_j\cdot\vec{r}.$$

On the other hand, $\vec{L}$ and any of its derivatives are clearly bounded at P. Then, by applying Leibniz's rule in order to differentiate the scalar product: $r^{m+k-2}\vec{L}\cdot\vec{r}$ m times w.r.t. r, we will obtain terms where r appears with an exponent of at least $k-1$. These terms will clearly vanish as we let $r\to 0$ and we will have:

$$\lim_{r\to 0}\frac{\partial^m}{\partial r^m}\left(r^{m+k-2}\vec{L}\cdot\vec{r}\right)=0.$$

Thus, by combining the two results presented above, we finally obtain:

$$\operatorname{Res}(\omega:P)=\frac{1}{m!(k-2)}\lim_{r\to 0}\frac{\partial^m}{\partial r^m}\left(r^{m+k-2}\vec{L}_\omega\cdot\vec{r}\right)$$

which concludes our proof. □

In closing this section, we will demonstrate how our approach can be adapted in $\mathbb{R}^2$ and we will exhibit the connection between ordinary residue calculus and our construction of form residues. Beffore proceeding, there is only one important remark that needs to be made and which concerns the special nature of the plane: namely, if $\omega\in\Omega^1(\mathbb{R}^2)$ then $\star\omega$ is also an element of $\Omega^1(\mathbb{R}^2)$. This means that ordinary path integrals and "surface" integrals are essentially the same thing and, eventhough this appears to be a rather subtle point, it leads to many important ramifications that will not be addressed in this paper.

For now, the construction of singular terms in the plane is essentially the same as before, with only one minor change in the definition of fundamental forms which must be modified as follows:

Definition 2.14. Consider a real array $T=[T_{j_1\cdots j_m}]$ of order m and dimension 2. The *fundamental form assigned to T* will be the 0-form of $\mathbb{R}^2\setminus\{O\}$:

$$\Omega_T=\begin{cases}-q\ln r & \text{if } m=0 \text{ and } T=[q],\\ \frac{T_{j_1\cdots j_m}x^{j_1}\cdots x^{j_m}}{r^{2m}} & \text{if } m\geq 1.\end{cases}\tag{2.13}$$

In a similar fashion, the *singular term associated to T* will be the 1-form defined on $\mathbb{R}^2\setminus\{O\}$ by:

$$P_T=-\star\,\mathrm{d}\Omega_T\tag{2.14}$$

The presence of the logarithm function in the above definition should be familiar to physicists and is due to the fact that $\mathrm{d}\ln r = \frac{1}{r^2}(x_1 \mathrm{d}x_1 + x_2 \mathrm{d}x_2)$, thus leading to the correct singular term of order 0. the rest of our results in this section hold in the planar case exactly as they hold in higher dimensional spaces, with the notable exception of the calculating proposition 2.13 which should read as follows:

Proposition 2.15. *Let A be an open ball of $\mathbb{R}^2$ and consider a 1-form ω defined and closed on A except for a point $P \in A$ where it presents a pole of order m. Then, the residue of ω at P is:*

$$\mathrm{Res}(\omega : P) = \frac{1}{m!} \lim_{r \to 0} \frac{\partial^m}{\partial r^m} \Big(r^m \vec{L}_\omega \cdot \vec{r} \Big) \tag{2.15}$$

where $\vec{L}_\omega = (\star^{-1}\omega)^\sharp$ and r is the usual polar coordinate, based on P.

Note: the above proposition differs from the higher dimensional case in the sense that it cannot be derived by 2.13 by merely setting $k = 2$ as that is not possible due to the $k - 2$ factor appearing in the denominator. However, the rest of our results may be trivially extended to the plane, simply by setting $k = 2$.

To see the connection between meromorphic functions and closed forms, let us consider a complex function $f(z) = u(z) + \mathrm{i}v(z)$ which is analytic on an open ball A of the complex plane $\mathbb{C}$ except, possibly, for a finite number of poles. If γ is a locally smooth curve of A - a 1-chain, to be exact - not passing through any of the poles of f, we will have:

$$\int_\gamma f(z)\mathrm{d}z = \int_\gamma u\mathrm{d}x - v\mathrm{d}y + \mathrm{i}\int_\gamma v\mathrm{d}x + u\mathrm{d}y = \int_\gamma \xi + \mathrm{i}\int_\gamma \eta$$

where $z = x + \mathrm{i}y$, $\xi = u\mathrm{d}x - v\mathrm{d}y$ and $\eta = v\mathrm{d}x + u\mathrm{d}y$. However, since f is analytic on A - except for a finite number of poles - the Cauchy-Riemann equations infer that: $u_x = v_y$ and $v_x = u_y$. Then, we shall have:

$$\mathrm{d}\xi = (u_y + v_x)\mathrm{d}y \wedge \mathrm{d}x = 0 \text{ and}$$
$$\mathrm{d}\eta = (v_y - u_x)\mathrm{d}y \wedge \mathrm{d}x = 0,$$

revealing ξ and η to be closed. Then, the integral of f over γ is reduced to the study of the integrals of ξ and η, which clearly fall within the domain of our previous results. Indeed, if Q is a pole of f, it is quite easy to show that:

$$\mathfrak{R}\Big[\mathrm{Res}(f : Q)\Big] = \mathrm{Res}(\eta : Q) \text{ and}$$
$$\mathfrak{I}\Big[\mathrm{Res}(f : Q)\Big] = -\mathrm{Res}(\xi : Q).$$

The relations above clearly demonstrate how ordinary complex residues can be identified with the form residues of 1-forms in the plane, and establish our main claim that Gauss's law and Cauchy's residue theorems can both be unified and given a new interpretation within the framework of deRham cohomology. Then, we can see that two distinct notions, such as the residue of a meromorphic function at one of its poles and the total electric charge accumulated at a singular point of an electric field are, in essence, the same thing, differing only in the setting where they are encountered. Ultimately, the multipole decomposition is much the same as Laurent decomposition with the $\frac{1}{(z-a)^k}$ term being interpreted as a 2^{k-1}-pole: monopoles correspond to the $1/(z - a)$ term and these terms generate the first deRham cohomology group of $\mathbb{C} \setminus \{a\}$.

However, we must stress that there *is* some fine print involved and we have only scratched the surface in the planar case. Indeed, 1-forms such as: $\omega = \frac{x\mathrm{d}x+y\mathrm{d}y}{r^2}$ - coming from applying d to fundamental forms and *not* $\star$d - are perfectly "singular" (with an intuitive interpretation of the term) and yet, they are not singular terms, neither can they be expressed as a series of singular terms. Fortunately, they have zero form residue and they do not interfere with proposition 2.15 but their very existence implies some very important ramifications. Their existence and nature seem to stem from the fact that the plane is unique as 1-forms and their $\star$ counterparts, (2-1)-forms, are the same. Still, we can only hint at their impact on our approach and their study may well form the core of a future paper on the matter.

3. Examples and Applications

This section is devoted in its entirety to examples and applications illustrating the use of the concepts we introduced. Bearing this in mind, our approach will not be very rigorous; instead, we will place emphasis on how the form residues actually work.

To begin with, we will perform a number of integrations in $\mathbb{R}^2$. Therefore, let us consider the following differential forms, which are defined on all of $\mathbb{R}^2 \setminus \{O\}$:

(1) $\xi = \frac{1}{(x^2+y^2)^2}\Big[(x^2 - y^2)\mathrm{d}y - 2xy\mathrm{d}x\Big]$
(2) $\omega = \frac{e^x}{x^2+y^2}\Big[(x\sin y - y\cos y)\mathrm{d}x + (x\cos y + y\sin y)\mathrm{d}y\Big]$

Furthermore, let C be the unit circle in $\mathbb{R}^2$ centered at the origin $O(0,0)$ and taken with the standard - anticlockwise - orientation[24]. We wish to evaluate the integrals $\int_C \rho$, $\int_C \xi$ and $\int_C \omega$.

Now, by a direct calculation, it can be shown that the forms above are closed in $\mathbb{R}^2 \setminus \{O\}$, that is: $\mathrm{d}\xi = \mathrm{d}\omega = 0$. In addition to that, one can easily see that they are closed with ω having a pole of order 0 at O while ξ has an order-1 pole at O; hence, we can apply propositions 2.8 and 2.15 in order to evaluate the desired integrals. Indeed, by proposition 2.8 we get:

$$\int_C - = 2\pi\ \mathrm{Ind}(C:O)\mathrm{Res}(-:O) = 2\pi\ \mathrm{Res}(-:O)$$

since $\mathrm{Ind}(C:O)$ is clearly $+1$. Consequently, we only need to evaluate the form residues at the origin, and this can be accomplished by means of proposition 2.15:

(1) Evaluation of $\mathrm{Res}(\xi : O)$
By applying proposition 2.15 to ξ - $m = 1$ - we get:

$$\vec{L}_\xi = \left(\star^{-1}\ \xi\right)^\sharp = \frac{1}{r^4}\Big[(x^2-y^2)\mathrm{d}x + 2xy\mathrm{d}y\Big]^\sharp = \frac{1}{r^4}\Big(x^2-y^2, 2xy\Big) \Rightarrow$$
$$r\vec{L}_\xi \cdot \vec{r} = \frac{1}{r^3}(x^3 - xy^2 + 2xy^2) = \frac{x}{r} = \alpha_1 \text{ [25]}$$

In polar coordinates, the result above is clearly independent of the radial coordinate r, that is: $\frac{\partial}{\partial r}\Big(r\vec{L}_\xi \cdot \vec{r}\Big) = 0$ and this leads to: $\mathrm{Res}(\xi : O) = 0$ [26]. Therefore, we have shown that: $\displaystyle\int_C \xi = 0$.

(2) Evaluation of $\mathrm{Res}(\omega : O)$:
Bearing in mind that ω has a pole of order 0 at O we obtain:

$$\vec{L}_\omega = \left(\star^{-1}\ \omega\right)^\sharp = \frac{e^x}{x^2+y^2}\Big(x\cos y + y\sin y, y\cos y - x\sin y\Big) \Rightarrow$$
$$\vec{L}_\omega \cdot \vec{r} = e^x \cos y$$

---

[24] In order to be absolutely clear on this point, a parametrization for the cycle we have in mind is: $x = \cos\theta$, $y = \sin\theta$ with $\theta \in [0, 2\pi]$ being the usual angular coordinate.

[25] Recall that $\alpha_j = \frac{x_j}{r}$ - see proofs of propositions 2.4 and 2.13.

[26] This should come as no surprise: ξ is just the singular term associated to the array $T = [-1 \quad 0]$.

Clearly, $r \to 0$ is equivalent to $\vec{r} = (x, y) \to (0, 0) = \vec{0}$. Therefore:

$$\lim_{r \to 0} \vec{L}_\omega \cdot \vec{r} = \lim_{\vec{r} \to \vec{0}} e^x \cos y = 1$$

Hence, $\mathrm{Res}(\omega : O) = 1$ which means that: $\displaystyle\int_C \omega = 2\pi$.

In the examples presented above, we have reduced integration to a straightforward algebraic calculation. We will further illustrate the process by a purposedly complex example in $\mathbb{R}^4$.

Indeed, let ψ be the differential 3-form defined on $\mathbb{R}^4 \setminus \{O\}$:

$$\begin{aligned}
\psi = \frac{1}{r^8}\Big[&\Big(2xr^4 - (w - 2x + y + z)r^2 - \\
&- 6x\big(w(w - x + y + z) + x^2 + (y - 2z)z - x(y + z)\big)\Big)\mathrm{d}z \wedge \mathrm{d}y \wedge \mathrm{d}w \\
&+ \Big(2yr^4 + (w - x + z)r^2 - \\
&- 6y\big(w(w - x + y + z) + x^2 + (y - 2z)z - x(y + z)\big)\Big)\mathrm{d}x \wedge \mathrm{d}z \wedge \mathrm{d}w \\
&+ \Big(2zr^4 + (w - x + y - 4z)r^2 + \\
&- 6z\big(w(w - x + y + z) + x^2 + (y - 2z)z - x(y + z)\big)\Big)\mathrm{d}y \wedge \mathrm{d}x \wedge \mathrm{d}w \\
&+ \Big(2wr^4 + (2w - x + y + z)r^2 - \\
&- 6w\big(w(w - x + y + z) + x^2 + (y - 2z)z - x(y + z)\big)\Big)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\Big]
\end{aligned}$$

Also, let Z denote the unit sphere S^3, centered at O and taken with the standard orientation with respect to the basis $\{\mathrm{d}x, \mathrm{d}y, \mathrm{d}z, \mathrm{d}w\}$. As in the previous examples, we wish to evaluate the integral $\displaystyle\int_Z \psi$. A direct calculation will reveal that ψ is closed in $\mathbb{R}^4 \setminus \{O\}$ and has a pole of order 2 at O. Therefore:

$$\int_Z \psi = (4 - 2) \cdot 2\pi^2 \,\mathrm{Res}(\psi : O)\mathrm{Ind}(Z : O) = 4\pi^2 \,\mathrm{Res}(\psi : O)$$ [27]

and we simply have to evaluate the residue of ψ at O by means of proposition 2.13.

So, by performing a few easy calculations, we obtain:

$$\begin{aligned}
r^4\vec{L}_\psi \cdot \vec{r} &= r^4(\star^{-1}\psi)^\sharp \cdot \vec{r} = \\
&= 2\Big(r^2 - \frac{2}{r^2}\big(2w^2 + (w - x + y + z)w + 3x^2 - x(y + z) + y(2y + z)\big) + 4\Big)
\end{aligned}$$

[27] Clearly, $\mathrm{Ind}(Z : O) = +1$.

Thus, by expressing this result in polar coordinates we get: $\frac{\partial^2}{\partial r^2}\left[r^4 \vec{L}_\psi \cdot \vec{r}\right] = 4$ [28] which yields: $\mathrm{Res}(\psi : O) = 1$. Hence: $\displaystyle\int_Z \psi = 4\pi^2$.

Note: Having computed the form residue at the origin for ξ, ω and ψ we can also deduce whether the forms are exact or not. As we saw, only ξ leaves zero residue at the origin and this means that ξ is the only exact form in the previous examples. This would be a rather hard problem to solve, but we have seen that it is readily reduced to a mere algebraic manipulation.

Now, a very important application of complex residue calculus lies in evaluating real improper integrals. In a similar fashion, we will see that the form residues which we introduced allow us to handle a large class of multiple - iterated - improper integrals[29]. Without further ado, we will proceed to a final series of examples illustrating the process.

Consider the k-fold integral: $\displaystyle\Im_k = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{\mathrm{d}x_1 \dots \mathrm{d}x_k}{\left(x_1^2 + \dots + x_k^2 + a^2\right)^{\frac{k+1}{2}}}$ with $a > 0$.

First, observe that in $\mathbb{R}^{k+1}$ the denominator - except for the (k+1) exponent - merely expresses the distance between a point $(x_1, \dots, x_k, 0)$ belonging to the k-dimensional hyperplane: $\pi_{k+1} = \langle x_1, \dots, x_k \rangle$ and the point $P = (0, \dots, 0, a)$. Furthermore, the spherical symmetry of the integrand and the presence of the (k+1) exponent - which equals the dimension of $\mathbb{R}^{k+1}$ - suggest that a singular term of order 0 and based at P lurks just around the corner. Thus, let us consider a monopole Φ based at P; its expression in Cartesian coordinates based at $O = (0, \dots, 0, 0)$ will be:

$$\Phi = \frac{x_{k+1} - a}{r_a^{k+1}}\, \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_k + \text{terms containing } \mathrm{dx_{k+1}}$$

where $r_a = \sqrt{x_1^2 + \dots + x_k^2 + (x_{k+1} - a)^2}$ denotes the distance between any given point $(x_1, \dots, x_{k+1})$ and P.

We will integrate Φ over the k-cycle Z_R which consists of:

(1) The section C_R of the k-sphere $S(P, R)$ - of radius R and centered at P - which lies "above" π_{k+1} - that is: $x_{k+1} \geq 0$ - and
(2) the disk D_R on π_{k+1} which is bounded by the intersection of $S(P, R)$ with π_{k+1}

The orientations given to C_R and D_R are such to ensure that Z_R is taken with the outward orientation. Bearing this in mind, we have: $\displaystyle\int_{Z_R} \Phi = \int_{D_R} \Phi + \int_{C_R} \Phi$.

Obviously, when we pull back and integrate Φ over $D_R \subseteq \pi_{k+1}$ where x_{k+1} is kept fixed to 0, the terms containing $\mathrm{d}x_{k+1}$ will vanish and $\int_{D_R} \Phi$ will be reduced to the

[28] Arguments of dimensional analysis can be used here: it is an easy tas to show that say $\frac{wx}{r^2}$ or any other dimensionless combination w.r.t. r will not depend on r.

[29] We will not go so far as to introduce concrete methods like the ones used in Complex Analysis. However, we hope that the number of examples provided will give the reader a fair notion of the potential of form residues.

following k-fold integral:

$$\int_{D_R} \Phi = a \int_{D_R} \frac{\mathrm{d}x_1 \ldots \mathrm{d}x_k}{\left(x_1^2 + \cdots + x_k^2 + a^2\right)^{\frac{k+1}{2}}}$$

Let $R \to \infty$. Clearly, $\lim_{R\to\infty} \int_{D_R} \Phi = a\mathfrak{I}_k$ since D_R takes up all of π_{k+1} as $R \to \infty$. Moreover, by computing the residue of Φ by means of proposition 2.13 we get: $\mathrm{Res}(\Phi : O) = \frac{1}{k-1}$ which yields - proposition 2.8: $\int_{Z_R} \Phi = A_k = 2\pi \frac{(\sqrt{\pi})^{k-1}}{\Gamma\left(\frac{k+1}{2}\right)}$ with A_k denoting the area spanned by the unit k-sphere S^k. This result is not modified by letting $R \to \infty$ and we obtain: $a\mathfrak{I}_k = A_k - \lim_{R\to\infty} \int_{C_R} \Phi$. But, $\int_{C_R} \Phi$ is the "solid angle" spanned by C_R[30] and it can be easily seen that as $R \to \infty$ this solid angle is just $\frac{1}{2}A_k$ since, for $R \to \infty$, C_R assumes the shape of a k-hemisphere - which clearly bounds half the surface spanned by a k-sphere of the same radius. Combining all of the above in a single relation, we obtain: $a\mathfrak{I}_k = \dfrac{1}{2}A_k$ which leads to our final result:

$$\mathfrak{I}_k = \frac{\pi}{a} \cdot \frac{(\sqrt{\pi})^{k-1}}{\Gamma\left(\frac{k+1}{2}\right)}$$

By considering singular terms of higher order we may evaluate other classes of k-fold integrals. Indeed, set $T = [0 \ldots 0 \quad 1]$ and let Ψ be the singular term associated to T and based at $(0, \ldots, 0, a) \in \mathbb{R}^{k+1}$ with $a > 0$. A direct computation will reveal that in Cartesian coordinates, Ψ can be expressed as follows:

$$\Psi = \left[(k+1)\frac{(x_{k+1} - a)^2}{r_a^{k+3}} - \frac{1}{r_a^{k+1}}\right]\mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_k + \text{terms containing } \mathrm{d}x_{k+1}$$

If Z_R, C_R, D_R are defined as in the previous example, it should be clear that: $\int_{Z_R} \Psi = 0$ since $\mathrm{Res}(\Psi : O) = 0$ - recall that Ψ is a singular term of order 1: it leaves no residue at P. In addition to that, when pulling back and integrating over D_R, the terms containing $\mathrm{d}x_{k+1}$ will vanish and $\int_{D_R} \Psi$ will be reduced to:

$$\int_{D_R} \Psi = \int_{D_R} \frac{x_1^2 + \cdots + x_k^2 - ka^2}{\left(x_1^2 + \cdots + x_k^2 + a^2\right)^{\frac{k+3}{2}}} \,\mathrm{d}x_1 \ldots \mathrm{d}x_k$$

However, if we let $R \to \infty$, it can be shown that the integral $\int_{C_R} \Psi$ vanishes. Indeed, from a dimensional standpoint, we can see that the exponent of the x_i's in the denominator of Ψ exceeds that of the numerator by k+1 and we are integrating over a k-cell: this will implement a scale factor of $\frac{1}{R}$ in $\int_{C_R} \Psi$ and, as $R \to \infty$, this

[30]Recall the proof of proposition 2.4.

will cause the integral to vanish[31]. This shows that for $R \to \infty$, $\int_{D_R}\Psi$ must vanish as well and, if we apply this result to the explicit expression of $\int_{D_R}\Psi$, we obtain:

$$\mathfrak{I}_k - (k+1)a^2 \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{\mathrm{d}x_1 \dots \mathrm{d}x_k}{\left(x_1^2 + \cdots + x_k^2 + a^2\right)^{\frac{k+3}{2}}} = 0$$

which, by substituting $\mathfrak{I}_k$, yields:

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{\mathrm{d}x_1 \dots \mathrm{d}x_k}{\left(x_1^2 + \cdots + x_k^2 + a^2\right)^{\frac{k+3}{2}}} = \frac{\pi}{(k+1)a^3} \cdot \frac{(\sqrt{\pi})^{k-1}}{\Gamma\left(\frac{k+1}{2}\right)}.$$

Naturally, the examples presented above are the simplest conceivable ones and, by merely playing around with the position of the multipole, we would be able to evaluate integrals with denominators far more complex. Furthermore, as should be obvious, the method of form residues can be applied to various trigonometric integrals as well, but, at this point, a simple method in the form of an "algorithm" - as in the case of Complex Analysis - cannot be presented, but could well constitute part of a future paper.

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[31] Of course, this "proof" lacks rigour. A rigorous argument along these lines is not hard to construct but, since we are interested in the overall method, and, in keeping with the "applied" style of this section, we will not venture into details.