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A paradox in optimal flow control of M/M/n queues

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Abstract

Optimal flow control problems of multiple-server (M/M/n) queueing systems are studied. Due to enhanced flexibility of the decision making, intuitively, we expect that grouping together separated systems into one system provides improved performance over the previously separated systems. This paper presents a result counter-intuitive against such an expectation. We consider a non-cooperative optimal flow control scheme M/M/n queueing systems where each of multiple players strives to optimize unilaterally its own power where the power of a player is the quotient of the throughput divided by the mean response time for the player. We report a counter-intuitive case where the power of every user degrades after grouping together $K (> 1)$ separated M/M/N systems into a single M/M/($K \times N$) system. Some numerical results are presented.

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1. Introduction

In computer and communication networks, flow control is one of the important means to utilize limited system resources effectively and to guarantee proper quality of service (QoS). Flow control adjusts the input flow (throughput) in order to provide good performance. Considering an optimal flow control problem, one may face the trade-off between the throughput and the response time. These two performance measures are mutually contradictory, that is, if one improves the system throughput, then the system response time degrades, and vice versa. Therefore, as the utility of each user (player), we use the power

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that is the quotient of the throughput divided by the average response time for the user (see, e.g., Giessler et al. [1], Kleinrock [23]).

We consider a system where multiple users (players) share an $M/M/n$ queueing system and where the utility of a player is the power. We can think of two typical performance optimization schemes: the *non-cooperative optimization* scheme and the *overall optimization* scheme. In the non-cooperative optimization scheme, each player strives to optimize unilaterally its own power given the decisions by others. In the overall optimization scheme, a single performance measure that is the total sum of the powers of all players is optimized by a single agent. The former is regarded as a non-cooperative game, and the equilibrium is called a *Nash equilibrium*. In this paper, we study the *Nash equilibrium* [4].

Douligeris and Mazumdar [5], and Zhang and Douligeris [6] studied algorithms to obtain Nash equilibria in flow control of $M/M/1$ queueing systems with multiple users. Their performance objective was to maximize the power. They proposed greedy algorithms, and showed convergence properties of them. State-dependent flow control was analyzed by Hsiao and Lazar [7], and Korilis and Lazar [8]. They considered a closed queueing network model, and maximized the average throughput subject to an upper bound on the average response time. In particular, Korilis and Lazar [8] derived the existence of equilibria using fixed-point theorems. Altman et al. [9] combined flow control and routing in a network model with several parallel links. Lazar [10] studied optimal flow control problems of an $M/M/n$ queue where one player maximizes the throughput subject to the constraint that the average time delay should not exceed a specified value. In this paper, we deal with flow control problems of $M/M/n$ queueing systems that have multiple players, where each player strives to optimize unilaterally its power.

Consider the operation of grouping together separated systems into one system. Due to enhanced flexibility in resource utilization, we expect that system grouping improves the system performance. For example, it has been shown analytically that grouping together separated systems improves the average response time although the utilization factor of each server remains the same (see [11]). Therefore, we expect performance improvement in terms of power by grouping together separate systems.

In noncooperative optimization of routing in networks and load balancing in distributed systems, however, it is known that the following phenomenon can occur: Grouping separated systems together and/or adding connections to the system may sometimes degrade the utilities for all users. The first example is the Braess paradox [12]. Other examples have been presented, for example, by Bean et al. [13], Calvert et al. [14], Cohen and Jeffries [15], Cohen and Kelly [16], Kameda et al. [17], Kameda and Pourtallier [18], Korilis et al. [1920], Roughgarden and Tardos [21]. However, most of the previous results are on routing (or load balancing) problems, and the paradox is seen in very limited models. Flow control is essentially different from them. In optimal flow control, it seems that no such paradoxes have been reported yet.

In this paper, we show that a paradox similar to the above ones may occur in noncooperative optimal flow control of $M/M/n$ queues. That is, we show a case where grouping leads to the degradation of the power to all players in noncooperative optimal flow control.

The rest of this paper is organized as follows. In Section 2, we describe a queueing system model and formulate overall and noncooperative optimal flow control schemes as nonlinear programming problems. In Section 3, we show the case of a paradox in noncooperative flow control of $M/M/n$ queueing systems. Finally, in Section 4, we conclude this paper. In the Appendix, we show the existence and the uniqueness of solutions to the optimization problems, and present algorithms that obtain the solutions.

2. The model and problem formulation

First, we consider an M/M/n queueing system. The system has an arrival stream of jobs that forms a Poisson process with rate Λ (jobs per unit time). Also, the system has n servers. The processing time of a job at each server is independent, identically distributed and according to an exponential distribution with mean $1/\mu$. Then, by denoting the traffic intensity by $\rho = \Lambda/(n\mu)$, the mean response time T of an arbitrary job is given by

$$T = T(\rho) = \begin{cases} \frac{B_n(\rho)}{n\mu(1-\rho)} + \frac{1}{\mu} & \text{if } \rho < 1, \\ \infty & \text{otherwise,} \end{cases} \tag{1}$$

for $l = 1, 2, \dots, r$, where

$$B_n(\rho) = \left[1 + \sum_{k=0}^{n-1} \frac{n!(1-\rho)}{k!(n\rho)^{n-k}} \right]^{-1} \tag{2}$$

is the probability that all servers are busy (called the *Erlang delay formula*). Next, we consider the M/M/n queueing system with r players. Each player sends the multiple-server, an arrival stream of jobs that is mutually independent and forms a Poisson arrival process with rate $\lambda_l, l = 1, 2, \dots, r$. Note that $\Lambda = \sum_l \lambda_l$ and that the processing time is independent of the player that sends the job. Then the mean response time T of an arbitrary job in this model is given by (1).

As the utility for each user in the flow control problems, we use the power for the user. Since the throughput is equivalent to the rate of the arrival flow (i.e., the arrival rate), from [3], the overall power P of the system and the power p_l of player l are given as

$$P = P(\Lambda) = \begin{cases} \frac{\Lambda}{T(\Lambda/(n\mu))} & \text{if } \Lambda \leq n\mu, \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

and

$$p_l = p_l(\lambda_1, \lambda_2, \dots, \lambda_r) = \begin{cases} \frac{\lambda_l}{T(\Lambda/(n\mu))} & \text{if } 0 \leq \lambda_l \leq n\mu - \sum_{j \neq l} \lambda_j, \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

$l = 1, 2, \dots, r$, respectively. Note that $\Lambda = \sum_l \lambda_l$. We have $P = \sum_l p_l$. P and p_l are positive for $\Lambda < n\mu$, and zero for $\Lambda = 0$ and $n\mu$.

We formulate, for the system described above, two typical optimal flow control schemes: the noncooperative optimization scheme (I) and the overall optimization scheme (II). The noncooperative and overall optimization schemes in an M/M/n queueing system with r players are presented as follows:

- (I) The noncooperative optimization scheme: Each player strives to maximize unilaterally its own power. That is, the noncooperative optimization is to find $\hat{\lambda}_l$ for each $l = 1, 2, \dots, r$, that satisfies

$$\hat{p}_l = \max_{\lambda_l \geq 0} p_l(\hat{\lambda}_1, \hat{\lambda}_2, \dots, \lambda_l, \dots, \hat{\lambda}_r).$$

As shown in our numerous experiments, we always observed symmetric solution to $\hat{\lambda}_l$ and \hat{p}_l , that is

$$\hat{\lambda}_l = \hat{\lambda} = \frac{\hat{\Lambda}}{r}, \quad \text{and} \quad \hat{p}_l = \hat{p}, \quad k = 1, 2, \dots, r. \tag{5}$$

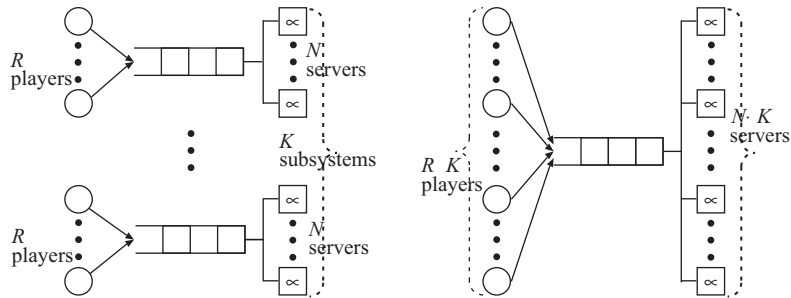


Fig. 1. [Left] A system consisting of K separated M/M/ N queues (S_I) vs. [Right] an M/M/($N \times K$) queue (S_U).

(II) The overall optimization scheme: A single agent maximizes the overall power of the system, that is, it strives to find $\tilde{\Lambda}$ that satisfies

$$\tilde{P} = P(\tilde{\Lambda}) = \max_{\Lambda \geq 0} P(\Lambda).$$

Then, the agent distributes equally to each player l the power \tilde{p}_l , and thus the throughput $\tilde{\lambda}_l$. Thus,

$$\tilde{p}_l = \tilde{p} = \frac{\tilde{P}}{r} \quad \text{and} \quad \tilde{\lambda}_l = \tilde{\lambda} = \frac{\tilde{\Lambda}}{r}. \tag{6}$$

In the appendix, we show existence and uniqueness of noncooperative and overall optimal solutions, and provide algorithms that obtain them.

Based on the above definitions, we consider two queueing system models as shown in Fig. 1:

- (1) One system, called S_I , consists of K subsystems each of which consists of a separated M/M/ N queue (N exponential servers with Poisson arrivals of jobs) and R independent players.
- (2) The other system, called S_U , results from grouping together all the above separated M/M/ N queues. That is, the system consists of one M/M/($K \times N$) queue and $K \times R$ independent players.

Obviously, we can formulate each subsystem of S_I and the system S_U using M/M/ n queues where $n = N$ and $r = R$ in S_I , and $n = K \times N$ and $r = K \times R$ in S_U , respectively. We express the parameters of a system by a quadruplet. That is, S_I given in (1) corresponds to (N, R, K, μ) , and S_U given in (2) corresponds to $(N \times K, R \times K, 1, \mu)$. We note that in S_I the power of users is independent of the value of K . Therefore, the solution of S_I for a given (N, R, K, μ) is identical to that of a system with parameters $(N, R, 1, \mu)$.

For both S_I and S_U , there are $K \times R$ players numbered by $(1, 1), (1, 2), \dots, (K, 1), \dots, (K, R)$. Denote the throughput of player (i, l) by λ_{il} . A subsystem k of S_I , has R players $(k, 1), (k, 2), \dots, (k, R)$. Note that for system S_U , we associate the symbols with the mark '.

From (5), in the noncooperative optimization scheme, we have for each subsystem k of S_I

$$\hat{\lambda}_{kl} = \hat{\lambda} = \frac{\hat{\Lambda}}{R} \quad \text{and} \quad \hat{p}_{kl} = \hat{p}, \quad l = 1, 2, \dots, R, \tag{7}$$

and for system S_U

$$\hat{\lambda}'_{il} = \hat{\lambda}' = \frac{\hat{\Lambda}'}{K \times R} \quad \text{and} \quad \hat{p}'_{il} = \hat{p}', \tag{8}$$

for $i = 1, 2, \dots, K, l = 1, 2, \dots, R$.

We define the degree of the paradox, δ , as follows:

$$\delta = \begin{cases} \frac{\hat{p} - \hat{p}'}{\hat{p}} & \text{if } \hat{p} > \hat{p}', \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Note that a paradox occurs when $\delta > 0$.

From (6), in the overall optimization scheme, we have for each subsystem k of S_I

$$\tilde{\lambda}_{kl} = \tilde{\lambda} = \frac{\tilde{\Lambda}}{R} \quad \text{and} \quad \tilde{p}_{kl} = \tilde{p} = \frac{\tilde{P}}{R}, \quad l = 1, 2, \dots, R, \tag{10}$$

and for system S_U

$$\tilde{\lambda}'_{il} = \tilde{\lambda}' = \frac{\tilde{\Lambda}'}{K \times R} \quad \text{and} \quad \tilde{p}'_{il} = \tilde{p}' = \frac{\tilde{P}'}{K \times R} \tag{11}$$

for $i = 1, 2, \dots, K, l = 1, 2, \dots, R$.

3. A case of a paradox

In this section, we show a case where a paradox occurs. Without loss of generality, we assume a time scale such that $\mu = 1$. We compare the two systems presented in Section 2 (Fig. 1): S_I —a system consisting of K separated M/M/ N queues (left) and S_U —an M/M/($N \times K$) queue (right). In the former, the flow control schemes are concerned with each separated M/M/ N queue. On the other hand, in the latter, the flow control schemes are concerned with the M/M/($K \times N$) system. We recall that the latter is the result of grouping together the former.

3.1. The results

In Table 1 and Figs. 2 and 3, we show the cases where $N = R = 11$ with the numbers N of servers and R of players in each subsystem.

Table 1 and Fig. 2 illustrate how the noncooperative optimal powers of each player in S_I and S_U depend on the number K of the subsystems grouped together. Fig. 2 shows a trend observed for the noncooperative optimization scheme. That is, in the scheme, the power of each player in S_U decreases as the number K of subsystems grouped together increases, for $N = 11$ and $R = 11$, as shown by the curve associated with the symbol S_U . On the other hand, naturally, the power of each player of each separated subsystem S_I remains constant regardless of the value of K as shown by the line associated with the symbol S_I . The trend looks counter-intuitive. Thus, we should not always expect that system grouping improves the system performance in the noncooperative flow control although we anticipate the opposite. As shown in Table 1, the degree of paradox increases in K .

Table 1

Power of each player and degree of paradox in the noncooperative optimization scheme, for $N = R = 11$ and various values of K

K	1	2	4	8	16
Power of each player	0.4248	0.4208	0.4135	0.4061	0.3999
Degree of paradox δ	0.000	0.009	0.027	0.044	0.059
K	32	64	128	256	512
Power of each player	0.3951	0.3915	0.3888	0.3868	0.3854
Degree of paradox δ	0.070	0.078	0.085	0.089	0.093

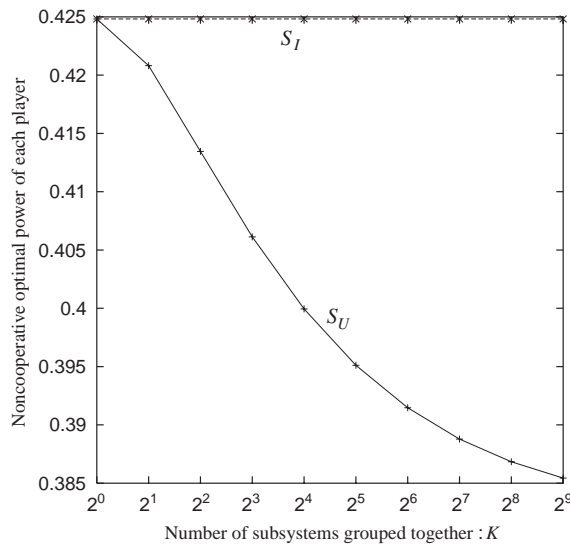


Fig. 2. Noncooperative optimization scheme.

On the other hand, Fig. 3 shows a naturally expected trend observed for the overall optimization scheme. That is, in the scheme, the power of each player in S_U increases as the number K of subsystems grouped together increases, for $N = 11$ and $R = 11$, as shown by the curve associated with the symbol S_U . On the other hand, naturally, the power of each player of each separated subsystem S_I remains constant regardless of the value of K as shown by the line associated with the symbol S_I .

3.2. Discussion of the results

To give some idea on the above-mentioned results, we present Fig. 4 for the case where $N = R = 1$. In the overall optimization scheme, each player always enjoys the increase in the power as the number K of subsystems grouped together increases. On the other hand, in the noncooperative optimization scheme, for the values of K from 1 till about 10, the power of each player increases although the improvement is smaller than that of the overall optimization scheme. For the values of K larger than 11, the power of each

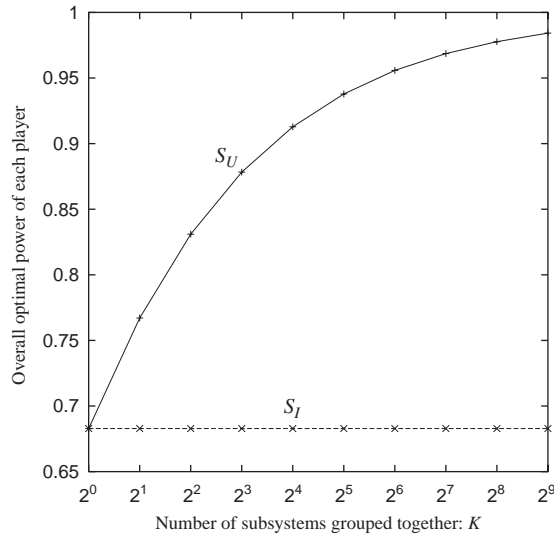


Fig. 3. Overall optimization scheme.

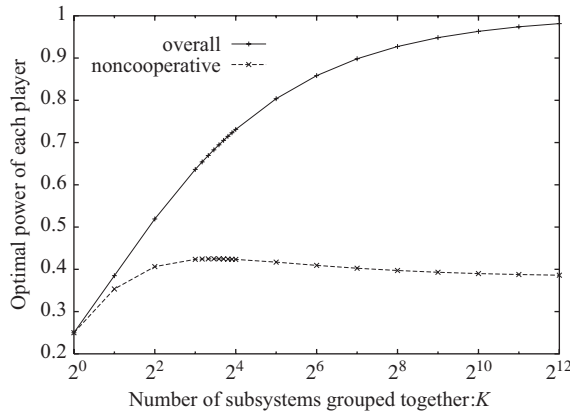


Fig. 4. Powers of each player in the noncooperative (\hat{p}) and overall optimization (\bar{p}) schemes as functions of the number K of subsystems grouped together, for $N = R = 1$.

player decreases as K increases, which shows the same trend as the above-mentioned paradoxical result. Note, from the definition, that the system S_U with parameters $(1, 1, \kappa, \mu)$ is identical with each subsystem of the system S_I with parameters $(\kappa, \kappa, 1, \mu)$ for any $\kappa \geq 1$. Therefore, each curve associated with the symbol S_U in Figs. 2 and 3 that starts at $K = 1$ with $R = N = 11$ shows the part of each corresponding curve in Fig. 4 for the domain of $K \geq 11$ with $R = N = 1$. In Fig. 4 ($R = N = 1$), we see that, in the noncooperative scheme, the power of each user is the largest for $K = 11$ (which is identical to that of a subsystem of S_I with $R = N = 11$) and decreases as K increases. Therefore, we see that if $R = N$, the paradox is the largest for the case with S_I with $R = N = 11$ and the value of K as large as possible.

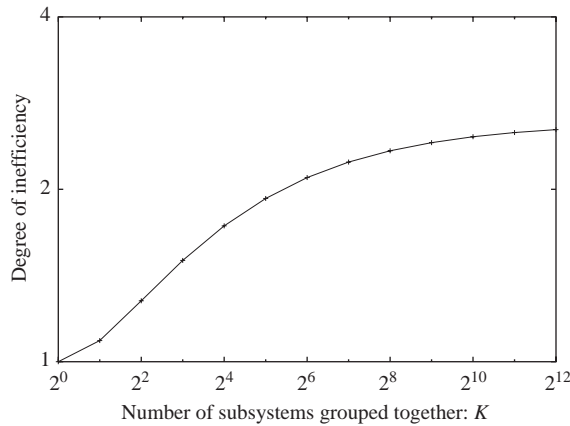


Fig. 5. Degree of inefficiency of the Nash equilibrium (\tilde{p}/\hat{p}), for $N = R = 1$.

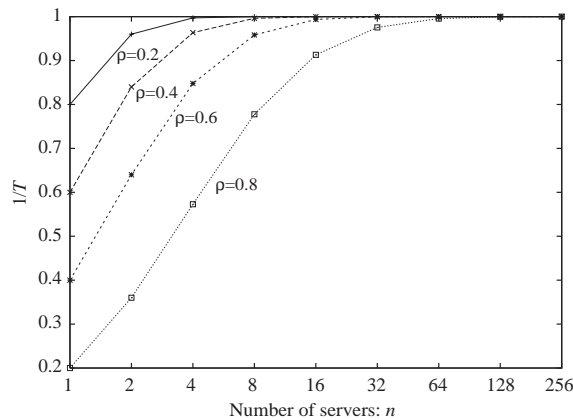


Fig. 6. [Effect of the number of servers on the mean response time of M/M/n]: The graph shows how $1/T$ (T is the mean response time given by (1)) increases as the number of servers increases with the traffic intensity ρ at each server remaining the same.

In the following, we seek the underlying structure of the system that may lead to the fact that the paradox is the largest for the case with S_I with $R = N = 11$ and the value of K as large as possible. We see in Fig. 5 that the degree of the inefficiency of the Nash equilibrium (\tilde{p}/\hat{p}) with $N = R = 1$ increases as K increases. That is, the possibility of paradox may start at some small value of K . We call this the effect A. On the other hand, we see in Fig. 6 that for small values of K , the grouping of servers improves the performance (responsiveness) of the system although the traffic intensity ρ of each server is kept identical, while this improvement is very small for large values of K . We call this effect B. We, therefore, see that effects A and B eliminate each other for small values of K whereas only the effect A is remarkable for the large values of K . Thus, as shown in Fig. 4, in the noncooperative optimization scheme with $N = R = 1$, the power is the largest for some intermediate value of K , that is in fact 11, and decreases as K increases. Therefore, we see that the paradox is the largest for the case with S_I with $R = N = 11$ and the value of K as large as possible.

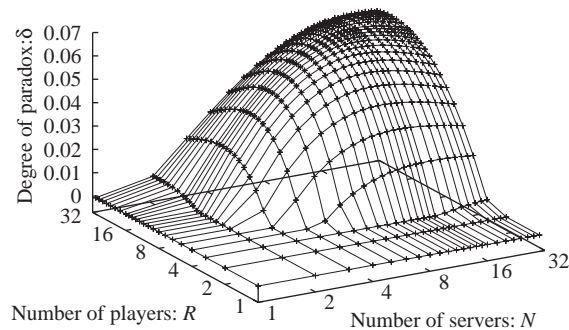


Fig. 7. [Noncooperative optimization scheme]: The graph shows the degree of paradox δ in noncooperative flow control for various combinations of N and R in the case where $K = 16$.

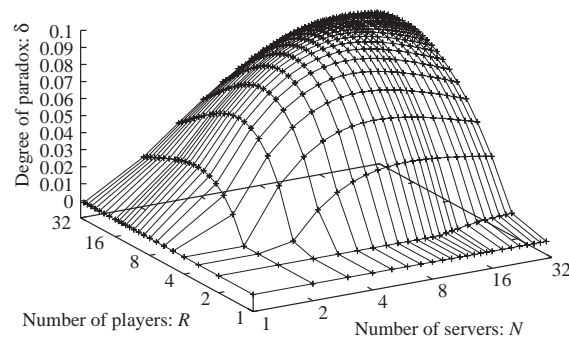


Fig. 8. [Noncooperative optimization scheme]: The graph shows the degree of paradox δ in noncooperative flow control for various combinations of N and R in the case where $K = 512$.

Furthermore, we have investigated the paradoxical behaviors exhaustively regarding the various combinations of the values of the number, N , of servers and the number, R , of players in each independent subsystem and the number, K , of subsystems to be grouped together. Figs. 7 and 8 show the degree of paradox for various combinations of the values of N and R in the two cases where $K = 16$ and 512, respectively. Also, we observe that the degree of paradox tends to be larger in the cases where the values of N is nearly equal to R than the other cases relevant to them. Thus, we observe again that the worst case of paradox may occur in the case where $N = R = 11$ and K has a large value. We have seen so far, however, no simple underlying structure that would support the fact that the worst case of paradox occurs in the case where $N = R$.

4. Conclusion

In this paper, we have studied the existence of a paradox in noncooperative flow control in M/M/ n queues. We have formulated the overall and noncooperative optimal flow control schemes of maximizing the power and have shown the existence and uniqueness of solutions. We have found a paradoxical

behavior in noncooperative optimal flow control similar to the Braess paradox of noncooperative optimal routing.

We have been interested in this research because optimal flow control may essentially be different from optimal routing and also because the paradox has been found to occur only in limited types of problems except optimal flow control. Our example of the paradox found in optimal flow control would suggest that the paradox could occur also in various other types of problems.

Appendix A. Existence and Uniqueness of Solutions and Algorithms

First, we show in this appendix, that there exist solutions to problems (I) and (II). Then, we prove that they are unique. Note that for M/M/1 queueing system models, closed form solutions to noncooperative and overall optimization problems have been obtained (see e.g., [5]). It seems, however, that there is no solution to M/M/n models for $n > 1$, has been published.

Lemma 1. (1) For $0 \leq \Lambda \leq n\mu$, the overall power, $P(\Lambda)$ is strictly concave in Λ .
 (2) For $0 \leq \lambda_l \leq n\mu - \sum_{j \neq l} \lambda_j$, the power of player l , $p_l(\lambda)$, is strictly concave in λ_l for $l = 1, 2, \dots, r$.

Proof. Since (1) is equivalent to (2) in the case where $r = 1$, we prove (2). The second derivative of p_l with respect to λ_l is as follows:

$$\begin{aligned} \frac{\partial^2 p_l(\lambda)}{\partial \lambda_l^2} &= 2 \frac{\partial}{\partial \lambda_l} \left\{ \frac{1}{T(\Lambda/(n\mu))} \right\} + \lambda_l \frac{\partial^2}{\partial \lambda_l^2} \left\{ \frac{1}{T(\Lambda/(n\mu))} \right\} \\ &= - \frac{2}{T(\Lambda/(n\mu))^2} \frac{\partial T(\Lambda/(n\mu))}{\partial \lambda_l} + \lambda_l \frac{\partial^2}{\partial \lambda_l^2} \left\{ \frac{1}{T(\Lambda/(n\mu))} \right\}. \end{aligned} \tag{A.1}$$

Now, we wish to show (A.1) is negative. From [22–24], the response time $T(\Lambda/(n\mu))$ is strictly increasing in ρ , where $\rho = \sum \lambda_l(n\mu) = \Lambda/(n\mu)$, and hence $T(\Lambda/(n\mu))$ is also strictly increasing in λ_l . Therefore, the first term of (A.1) is negative.

We rewrite the second term of (A.1) as follows:

$$\lambda_l \frac{\partial^2}{\partial \lambda_l^2} \left\{ \frac{1}{T(\Lambda/(n\mu))} \right\} = \lambda_l \frac{\partial^2}{\partial \rho^2} \left\{ \frac{1}{T(\Lambda/(n\mu))} \right\} \left(\frac{\partial \rho}{\partial \lambda_l} \right)^2.$$

From [25], we have $d^2 T(\Lambda/(n\mu))^{-1} / d\Lambda^2 < 0$. Note that $\partial e / \partial \lambda_l = 1/(n\mu)$. Therefore, the second term of (A.1) is nonpositive. Finally, the left-hand side of (A.1) is negative. \square

We define the function g by

$$g(\Lambda) = \frac{T(\Lambda/(n\mu))}{T'(\Lambda/(n\mu))} - \frac{\Lambda}{r}, \tag{A.2}$$

where $T'(\Lambda/(n\mu)) = dT(\Lambda/(n\mu))/d\Lambda$.

Lemma 2. Any solution to the noncooperative optimization problem (I) is symmetric (i.e., $\hat{\lambda}_l = \hat{\lambda}_{l'} = \hat{\Lambda}/r$, $l, l' = 1, 2, \dots, r$), and satisfies $g(\hat{\Lambda}) = 0$.

Proof. We denote a solution to (I) by $\hat{\lambda}$. It satisfies $\hat{\lambda}_l > 0$ for all $l=1, 2, \dots, r$ and $\sum_l \hat{\lambda}_l < n\mu$. Moreover, it is a solution to $\partial pl(\lambda)/\partial \lambda_l = 0, l = 1, 2, \dots, r$, which is equivalent to

$$\lambda_l \frac{\partial T(\Lambda/(n\mu))}{\partial \lambda_l} - T(\Lambda/(n\mu)) = 0, l = 1, 2, \dots, r. \quad (\text{A.3})$$

Since $T(\Lambda/(n\mu))$ depends only on Λ , then $\partial T(\Lambda/(n\mu))/\partial \lambda_l = \partial T(\Lambda/(n\mu))/\partial \lambda_{l'} = dT(\Lambda/(n\mu))/d\Lambda, l, l' = 1, 2, \dots, r$. Then, any solution to the problem (I) satisfies $\lambda_l = T(\Lambda/(n\mu))/T'(\Lambda/(n\mu))$ for all $l = 1, 2, \dots, r$ with $T'(\Lambda/(n\mu)) = dT(\Lambda/(n\mu))/d\Lambda$. $\hat{\lambda}$ is therefore symmetric, and given by $g(\hat{\Lambda}) = 0$ and $\hat{\lambda}_l = \hat{\Lambda}/r, l = 1, 2, \dots, r$. \square

Therefore, we obtain the following proposition:

Proposition 3. (A) *The solution to the noncooperative optimization problem (I) is unique, and is a solution of the following equation:*

$$g(\Lambda) = \frac{T(\Lambda/(n\mu))}{T'(\Lambda/(n\mu))} - \frac{\Lambda}{r} = 0, \quad (\text{A.4})$$

(B) *The solution to the overall optimization problem (II) is unique, and is a solution of the following equation:*

$$\frac{T(\Lambda/(n\mu))}{T'(\Lambda/(n\mu))} - \Lambda = 0. \quad (\text{A.5})$$

Proof. (A) From Lemma 2, a solution to problem (I) is given by $g(\Lambda) = 0$, which is (A.4). If g is a strictly decreasing function of Λ for $0 \leq \Lambda \leq n\mu$, and satisfies $g(0) > 0$ and $g(n\mu) < 0$, there exists a unique solution to (A.4). The derivative of g with respect to Λ is

$$\begin{aligned} \frac{dg(\Lambda)}{d\Lambda} &= \frac{T'(\Lambda/(n\mu))^2 - T(\Lambda/(n\mu))T''(\Lambda/(n\mu))}{T'(\Lambda/(n\mu))^2} - \frac{1}{r} \\ &= -\frac{T(\Lambda/(n\mu))T''(\Lambda/(n\mu)) - 2T'(\Lambda/(n\mu))^2}{T'(\Lambda/(n\mu))^2} - 1 - \frac{1}{r}, \end{aligned}$$

where $T''(\Lambda/(n\mu)) = d^2T(\Lambda/(n\mu))/d\Lambda^2$. In [25], it is shown that $d^2T^{-1}/d\Lambda^2 < 0$, that is to say that $T(\Lambda/(n\mu))T''(\Lambda/(n\mu)) - 2T'(\Lambda/(n\mu))^2 > 0$. Then, we have $dg(\Lambda)/d\Lambda < 0$ for $0 \leq \Lambda \leq n\mu$. Therefore, g is a strictly decreasing function of Λ .

Next, we show $g(0) > 0$ and $g(n\mu) < 0$. From [22–24], the function T is increasing and strictly convex in Λ for $0 \leq \Lambda \leq n\mu$. Then $T'(\Lambda/(n\mu))$ is positive for $\Lambda = 0$. From (1), $T(\Lambda/(n\mu))$ is also positive for $\Lambda = 0$. Therefore, from (A.2) we have $g(0) > 0$.

The derivatives of T and B_n are

$$\frac{dT(\Lambda/(n\mu))}{d\Lambda} = \frac{B'_n(\Lambda/(n\mu))(n\mu - \Lambda) + B_n(\Lambda/(n\mu))}{(n\mu - \Lambda)^2}$$

and

$$B'_n(\lambda/(n\mu)) = \frac{dB_n(\lambda/(n\mu))}{d\lambda} \\ = \frac{B_n(\lambda/(n\mu)) [n\mu\lambda(1 - B_n(\lambda/(n\mu))) + n(n\mu - \lambda)^2]}{n\mu\lambda(n\mu - \lambda)},$$

respectively. Then if $\lambda \neq 0$, $g(\lambda)$ is rewritten as follows:

$$g(\lambda) = \frac{n\lambda(n\mu - \lambda)(\mu B_n(\lambda/(n\mu)) + n\mu - \lambda)}{B_n(\lambda/(n\mu)) [n\mu\lambda + (1 - B_n(\lambda/(n\mu))) + n(n\mu - \lambda)^2]} - \frac{\lambda}{r}.$$

Since $B_n(1)=1$, we obtain $g(n\mu) < 0$. Therefore, (A.4) has a unique solution.

(B) We denote a solution to (II) by $\tilde{\lambda}$. Note that P is positive for $0 < \lambda < n\mu$, and zero for $\lambda \leq 0$ and $\lambda \geq n\mu$. Also from Lemma 1, P is strictly concave for $0 \leq \lambda \leq n\mu$. Then problem (II) has a unique solution, and $\tilde{\lambda}$ is a solution to $dP(\lambda)/d\lambda = 0$ for $0 < \tilde{\lambda} < n\mu$, which is equivalent to (A.5). \square

Each of Eqs. (A.4) and (A.5) has a single variable, respectively. Therefore, we can obtain the solutions simply by using an iterative algorithm. We provide such an algorithm.

Algorithm

Step 1: Set $x_0 = 0$, $y_0 = n\mu$, $\varepsilon_0 = y_0 - x_0$, $\varepsilon > 0$, and $k = 0$.

Step 2: (I) if $T(\lambda/(n\mu))/T'(\lambda/(n\mu)) - \frac{\lambda}{r} \leq 0$, then set $y_k = \varepsilon_k$ otherwise set $x_k = \varepsilon_k$.

(II) if $T(\lambda/(n\mu))/T'(\lambda/(n\mu)) - \lambda \leq 0$, then set $y_k = \varepsilon_k$ otherwise set $x_k = \varepsilon_k$.

Step 3: Compute $\varepsilon_k = y_k - x_k$.

Step 4: If $|\varepsilon_k| > \varepsilon$ then go to step 2.

Step 5: (I) Set $\hat{\lambda} = \varepsilon_k$.

(II) Set $\tilde{\lambda} = \varepsilon_k$.

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Further Reading

- [26] Rosen JB. Existence and uniqueness of equilibrium points for concave N-person games. *Econometrica* 1965;33:153–63.